Chapter 8 Flows and Lie Bracket

- 8.1 Integral curves *(covered last week)*
- 8.2 Flow of a vector field *(covered last week)*
- 8.3 Lie bracket (covered last week)
- 8.4 Lie derivative
- 8.5 Lie algebra of a Lie group *(EXTRA)*
- 8.6 The Frobenius theorem *(EXTRA)*

8.1 Integral curves

Let M be a smooth manifold and X a smooth vector field on M.

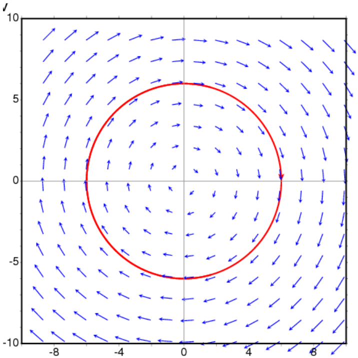
Definition

A smooth curve $\gamma : I \subseteq\longrightarrow M$ is called an ***integral curve*** of X if, for all $t \in I$:

$$\gamma'(t) = X_{\gamma(t)} \ .$$

Example 1. Consider the constant vector field $Y = \frac{\partial}{\partial y}$ in $\mathbb{R}^2$.
$\gamma(t) = (x(t), y(t))$ is an integral curve $\Leftrightarrow (x'(t), y'(t)) = (0, 1)$.
Integral curves: $\gamma(t) = (x_0, t + y_0)$. Integral curves = vertical lines.

Example 2. Consider the vector field $X = -y\frac{\partial}{\partial y} + x\frac{\partial}{\partial y}$.
Integral curves = Circles centered at the origin. Exercise.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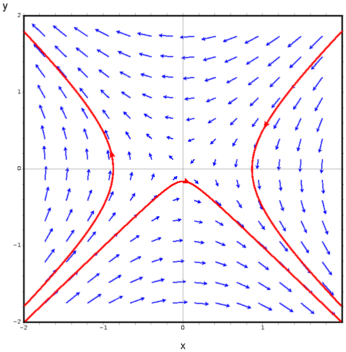
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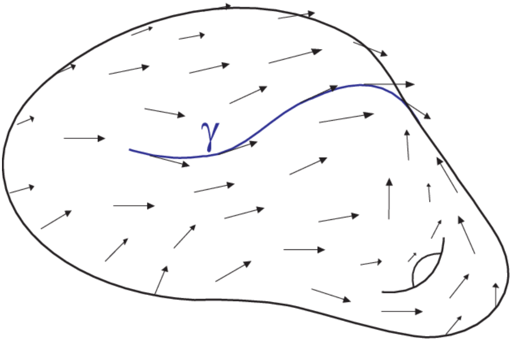

γ

Theorem

There exists a unique integral curve through any point.

More precisely: $\forall p \in M$, $\exists!$ *integral curve* $\gamma \colon I \to M$, *with* I *maximal, s.t.* $\gamma(0) = p$.

Proof. If $U \subseteq \mathbb{R}^m$, X is a given by $F \colon U \to \mathbb{R}^m$, and the equation of an integral curve is $\gamma'(t) = F(\gamma(t))$. Conclude by the Picard-Lindelöf (i.e. Cauchy-Lipschitz) theorem.

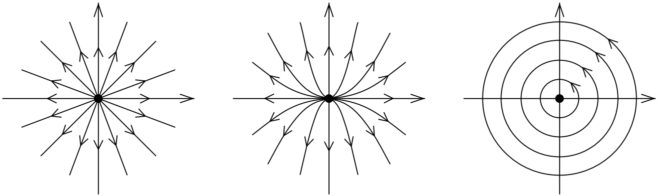
In general, use charts to apply with previous result in local charts. This shows local existence and uniqueness, and conclude.

Remark. Prerequisite: Basic theory of ODEs. Reference: [Lee, Appendix D].
In addition to the existence and uniqueness result, we have:

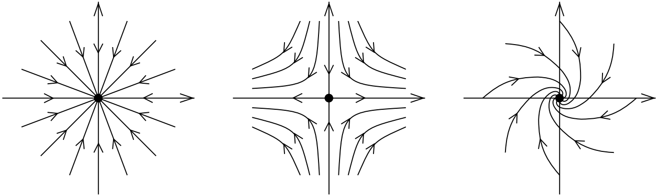
- Smooth dependence of solutions of ODEs on initial condition.
- If the maximal interval I has a finite bound, for instance $I = (a, b)$ with $b < +\infty$, then $\gamma(t)$ leaves every compact set when $t \to b^-$.

  In particular, if M is compact, then $I = \mathbb{R}$: all integral curves are ***complete***, i.e. X is complete.

Remark. If $X|_p = 0$, then the constant curve $\gamma(t) = p$ is an integral curve. p is called a ***zero*** or a ***singular point*** of the vector field.



$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $\quad$ $x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ $\quad$ $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$



$-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ $\quad$ $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ $\quad$ $(y-x)\frac{\partial}{\partial x} - (y+x)\frac{\partial}{\partial y}$

8.2 Flow of a vector field

Let M be a smooth manifold and X a smooth vector field on M.

For any $p \in M$, denote $\varphi_t^X(p) := \gamma(t)$, where γ is the integral curve of X through p.

Remark. A priori, $\varphi_t^X(p)$ is only well-defined for t sufficiently small.

Theorem

- *The map $\mathbb{R} \times M \to M$, $(t, p) \mapsto \varphi_t^X(p)$ is smooth on its domain of definition (which is a neighborhood of $\{0\} \times M$).*
- $\varphi_s^X \circ \varphi_t^X(p) = \varphi_{t+s}^X(p)$ *whenever well-defined.*

Any map $\mathbb{R} \times M \to M$ as in the theorem is called a smooth ***flow*** on M.

Proof.

- Smooth dependence of solution of an ODE on initial condition.
- If γ is the integral curve through p, then so is $\gamma(t_0 + t)$ is the integral curve through $\gamma(t_0)$.

Corollary

- *$\varphi_0^X : M \to M$ is the identity map.*
- *If φ_t^X is well-defined, then it is a diffeomorphism of M with inverse φ_{-t}^X.*
- *If well-defined, the map $t \mapsto \varphi_t^X$ is a group homomorphism $\mathbb{R} \to \mathrm{Diff}(M)$.*

Terminology. The flow is called ***complete*** if it is defined on $\mathbb{R} \times M$, i.e. all integral curves are complete (defined on $\mathbb{R}$), i.e. X is a ***complete vector field***.

Fact. If M is compact, any vector field on M is complete.

Example. Let $X = \frac{\partial}{\partial y}$ on $M = \mathbb{R}^2$. Then $\varphi_t^X(x, y) = (x, y + t)$.

Exercise. Let $M = \mathbb{R}^2 - \{0\}$. Find two vector fields whose integral curves are rays emanating from the origin, one complete, the other incomplete.

A normal form theorem:

Theorem

Let X be a smooth vector field on M. If $p \in M$ is regular (i.e. nonsingular) point, then there exists local coordinates $x^1, \ldots, x^p$ near p s.t. $X = \frac{\partial}{\partial x^1}$.

Remark. For the proof of this theorem and more details on flows, refer to Lee's book.

8.3 The Lie bracket

Recall that a *derivation* on an algebra A is a $\mathbb{R}$-linear map $D\colon A \longrightarrow A$ s.t.

$$D(fg) = D(f)\,g + f\,D(g)$$

Fact. If D_1 and D_2 are derivations, then $D := D_1 \circ D_2 - D_2 \circ D_1$ is a derivation.

Proof. Stupid algebra computation. Do it!!

Definition

D is denoted $[D_1, D_2]$ and called the ***Lie bracket*** (or *commutator*) of D_1 and D_2.

Recall that there is a bijection between smooth vector fields and derivations on a smooth manifold, more precisely there is a linear isomorphism

$$\begin{aligned}\Gamma(\mathrm{T}M) &\to \{\text{Derivations on } C^\infty(M,\mathbb{R})\} \\ X &\mapsto (f \mapsto X \cdot f)\end{aligned}$$

With this correspondence, we get the ***Lie bracket*** of vector fields:

Proposition

If X and Y are smooth vector fields on M, there exists a unique smooth vector field $[X, Y]$ such that for any smooth function $f : M \to \mathbb{R}$,

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot g) .$$

Proposition (Properties of the Lie bracket)

- $[\cdot,\cdot]$ *is* $\mathbb{R}$*-bilinear:* $[\lambda X_1 + \mu X_2, Y] = \dots$ *and* $[X, \lambda Y_1 + \mu Y_2] = \dots$
- $[\cdot,\cdot]$ *is antisymmetric:* $[Y,X] = -[X,Y]$*. In part.* $[X,X] = 0$*.*
- *Jacobi identity:* $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$*.*

Proof. Idiotic algebra computations. Do it!

Definition

A vector space A equipped with a bilinear map $[\cdot,\cdot] : A \times A \to A$ satisfying the properties above is called a ***Lie algebra***.

Examples.

1. $[\cdot,\cdot] = 0$. *(abelian Lie algebra)*
2. $\{$Derivations on $C^\infty(M,\mathbb{R})\}$
3. $\Gamma(\mathrm{T}M)$
4. Lie algebra of a Lie group (see later).

Proposition (Further properties of the Lie bracket)

- $[fX, Y] = f[X, Y] - (Y \cdot f)X.$
- *Naturality of the Lie bracket:* $f_*[X, Y] = [f_*X, f_*Y].$

Proof. Moronic algebra computations. Do it!

Proposition (Lie bracket in coordinates)

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

$$\left[\sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}, \sum_{j=1}^{m} Y^j \frac{\partial}{\partial x^j}\right] = \sum_{i,j=1}^{m} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$$

8.4 The Lie derivative

Let M be a smooth manifold and $X \in \Gamma(\mathrm{T}M)$. Assume that for $t > 0$ sufficiently small, the flow $\varphi_t^X := \varphi_t \in \mathrm{Diff}(M)$ is well-defined.

Lie derivative of a function

For any $f \in C^\infty(M, \mathbb{R})$, the ***pullback*** of f by φ_t is the function $(\varphi_t)^* f \in C^\infty(M, \mathbb{R})$ defined by $(\varphi_t)^* f := f \circ \varphi_t$.

Definition
The ***Lie derivative*** of $f \in C^\infty(M, \mathbb{R})$ with respect to X is the function $\mathcal{L}_X f \in C^\infty(M, \mathbb{R})$ defined by

$$\mathcal{L}_X f := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} (\varphi_t)^* f$$

Proposition

$\mathcal{L}_X f$ is well-defined and $\mathcal{L}_X f = X \cdot f$ (i.e. $\mathcal{L}_X = \mathrm{d}f(X)$).

Proof. Let us prove the identity pointwise. Let $p \in M$.

For $t > 0$ sufficiently small, $(\varphi_t)^* f(p) = f \circ \varphi_t(p)$ is well-defined.

Moreover, this is a smooth function of t, and by definition of the differential $\mathrm{d}f$:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|_{t=0}} f \circ \varphi_t(p) = (\mathrm{d}f)_{|_{\varphi_0(p)}} \left(\frac{\mathrm{d}}{\mathrm{d}t}_{|_{t=0}} \varphi_t(p) \right) = (\mathrm{d}f)_{|_p} \left(X_p \right) .$$

End of proof.

Lie derivative of a vector field

For any $Y \in \Gamma(\mathrm{T}M)$, the ***pullback*** of Y by φ_t is the vector field $(\varphi_t)^* Y \in \Gamma(\mathrm{T}M)$ defined by $(\varphi_t)^* Y := \left(\varphi_t^{-1}\right)_* Y$.

Definition

The ***Lie derivative*** of Y w.r.t. X is the vector field $\mathcal{L}_X Y \in \Gamma(\mathrm{T}M)$ defined by

$$\mathcal{L}_X Y := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} (\varphi_t)^* Y \, .$$

Proposition

$\mathcal{L}_X Y$ is well-defined and $\mathcal{L}_X Y = [X, Y]$.

Proof. Let us work in a local chart U. By naturality of the Lie bracket and the Lie derivative, it is enough to do the case $U \subseteq \mathbb{R}^m$.

Hence we can assume that X and Y are both smooth maps $U \longrightarrow \mathbb{R}^m$.

We compute:

$$\begin{aligned}
\left[(\varphi_t)^* Y\right]_{|_p} &= \left[(\varphi_t^{-1})_* Y\right]_{|_p} \\
&= (\mathrm{d}\varphi_t^{-1})_{|_{\varphi_t(p)}}(Y_{|_{\varphi_t(p)}}) \\
&= (\mathrm{d}\varphi_{t\,|_p})^{-1}\,(Y(\varphi_t(p)))
\end{aligned}$$

Taking the derivative at $t = 0$:

$$\begin{aligned}
(\mathcal{L}_X Y)(p) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}_{|_{t=0}} (\mathrm{d}\varphi_{t\,|_p})^{-1}\right) Y(p) + \frac{\mathrm{d}}{\mathrm{d}t}_{|_{t=0}} Y(\varphi_t(p)) \\
&= -\,\mathrm{d}X_{|_p}(Y(p)) + \mathrm{d}Y_{|p}(X(p)) \\
&= -\partial_Y X(p) + \partial_X Y(p) \\
&= [X, Y](p)\,.
\end{aligned}$$

8.5 Lie groups and Lie algebras (EXTRA)

Let G be a lie group.

For any $g \in G$, the left- and right- multiplication by g are smooth diffeomorphisms $L_g : G \to G$ and $R_g : G \to G$.

A vector field $X \in \Gamma(\mathrm{T}\,G)$ is called ***left-invariant*** if $(L_g)_* X = X$ for all $g \in G$.

A left-invariant vector-field X is uniquely determined by $X_{|_e}$. More precisely:

Lemma
$X \mapsto X_{|_e}$ is a linear isom. between the space of left-invariant vector fields and $\mathrm{T}_e\,G$.

Moreover, the subspace of $\Gamma(\mathrm{T}\,G)$ of left-invariant vector fields is stable under $[\cdot, \cdot]$.

Therefore, we can put a structure of Lie algebra on $\mathrm{T}_e\,G$ using the isomorphism above. This is the ***Lie algebra of*** G, denoted $\mathrm{Lie}(G)$ of $\mathfrak{g}$.

Theorem (PSEUDO theorem)

There is more or less a bijective correspondence between finite-dimensional Lie algebras and Lie algebras of finite-dimensional Lie groups.

8.6 The Frobenius theorem (EXTRA)

Definition

Let $D \subseteq \mathrm{T}M$ be a subbundle. This is called a ***distribution*** on M.

- D is called ***involutive*** if it is stable under the Lie bracket.
- D is called ***integrable*** if for all $p \in D$, $\exists$ a submanifold $N \subseteq M$ s.t. $D_p = \mathrm{T}_p N$.

Theorem (Frobenius theorem)

D is integrable if and only if D is involutive.

Examples.

- 0*-dimensional distributions,* 1*-dimensional, and* m*-dimensional distributions are integrable.*
- *Tangent spaces to foliations (e.g. fiber bundles) are integrable.*
- *completely non-integrable distributions: sub-Riemannian manifolds.*