Students evaluation for the “Manifolds” course (do it **before tomorrow!**):
http://evaluation.tu-darmstadt.de/evasys/online.php?pswd=Y5DJH

Chapter 7 Vector fields

7.1 Definition

Definition

Let $\pi: E \longrightarrow B$ be a fiber bundle. A ***section of*** E is a map $s: B \longrightarrow E$ s.t. $\pi \circ s = \mathrm{id}_B$.
The space of sections of E is denoted $\Gamma(E)$.

Remarks.

- $s: B \longrightarrow E$ is a section $\Leftrightarrow \forall x \in B\ \ s(x) \in E_x$, where $E_x: \pi^{-1}(x)$ is the fiber over x.
- Henceforth, we only consider smooth fiber bundles and smooth sections.

Definition

Let M be a smooth manifold. A (smooth) ***vector field*** on M is a section of $\mathrm{T}\,M$.
The space of vector fields on M is denoted $\Gamma(\mathrm{T}\,M)$.

Remarks

- A vector field on M is a smooth map $X: M \longrightarrow \mathrm{T}\,M$ s.t. $\forall p \in M, X(p) \in \mathrm{T}_p\,M$.
- We write X_p (or $X_{|_p}$) instead of X_p.

Examples.

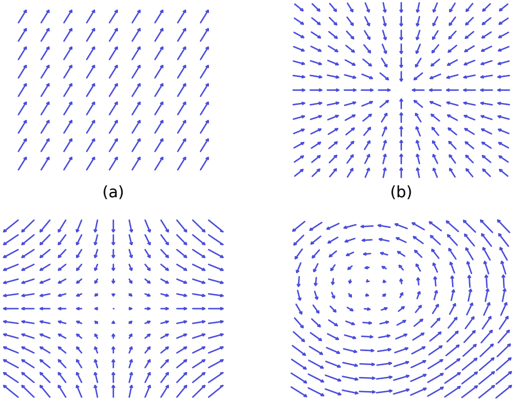
Example 1. Let $F\colon U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a smooth function.
Define $X\colon U \to \mathrm{T}\,U$ by $X_p = F(p) \in \mathbb{R}^m \approx \mathrm{T}_p\,U$. Then X is a vector field on U.

Remark. In fact, a vector field is always of this form, when looking at it in a chart.

For instance, take $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

(a) $F(x, y) = (1, 2)$
(b) $F(x, y) = (-x, -y)/\sqrt{x^2 + y^2}$
(c) $F(x, y) = (\cos y, \sin x)$
(d) $F(x, y) = (x, -y)$

Examples.

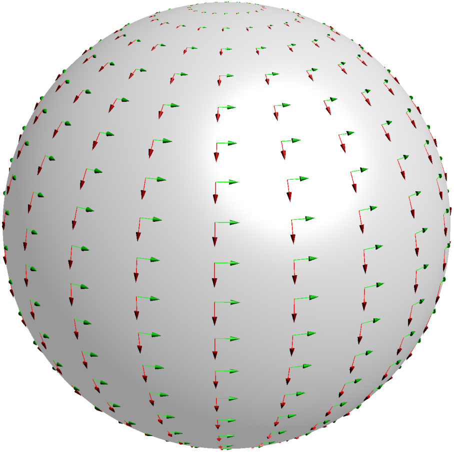


(a) (b) (c) (d)

Examples.

Example. Two vector fields on the sphere:

- $X_{(x,y,z)} = (-y, x, 0)$
- $Y_{(x,y,z)} = (xz, yz, -x^2 - y^2)$



Examples.

Example: Coordinate vector fields.

Let $(x^1, \ldots, x^m)$ be local coordinates on $U \subseteq M$.
Then $\dfrac{\partial}{\partial x^1}, \ldots, \dfrac{\partial}{\partial x^m}$ are vector fields on U, called ***coordinate vector fields***.

7.2 Vector fields as derivations

Let X be a vector field on a smooth manifold M. Let $f\colon M \to \mathbb{R}$ be a smooth function.

For every $p \in M$, we can define $\mathrm{d}f_p(X_p) \in \mathbb{R}$.
The function $\mathrm{d}f(X)\colon p \mapsto \mathrm{d}f_p(X_p)$ is a smooth function $M \to \mathbb{R}$.

Recall that a tangent vector $X_p \in \mathrm{T}_p M$ can also be seen as derivation on $C^\infty(M, \mathbb{R})$.
With this point of view, the function $\mathrm{d}f(X)$ is alternatively denoted $\frac{\partial}{\partial X} f$ or $X(f)$ or $X \cdot f$.

Definition

A ***derivation*** on $A := C^\infty(M, \mathbb{R})$ is a $\mathbb{R}$-linear map $D\colon A \to A$ s.t. $\forall f, g \in A$:

$$D(fg) = D(f)\, g + f\, D(g) \qquad \textit{(Leibniz rule)}$$

Proposition

We have a linear isomorphism:

$$\begin{aligned} \Gamma(\mathrm{T} M) &\to \{\textit{Derivations on } C^\infty(M, \mathbb{R})\} \\ X &\mapsto \frac{\partial}{\partial X} \end{aligned}$$

7.3 Pushforward of a vector field

Let X be a vector field on a smooth manifold M.
Let $f\colon M \longrightarrow N$ be a smooth function.

For every $p \in M$, we can define $\mathrm{d}f_p(X_p) \in \mathrm{T}_{f(p)} \in \mathrm{T}_{f(p)}\, N$.
We have a smooth map $\mathrm{d}f(X)\colon M \longrightarrow \mathrm{T}\, N$.

If f is a diffeo, consider $Y\colon N \longrightarrow \mathrm{T}\, N$ defined by $Y = \mathrm{d}f(X) \circ f^{-1}$.

Proposition

Y is a smooth vector field on N, called ***pushforward*** *of X by f and denoted f_*X.*

Proposition

Let $f : M \to N$ be a diffeo. The pushforward map

$$\begin{aligned} f_* : \Gamma(\mathrm{T}M) &\to \Gamma(\mathrm{T}N) \\ X &\mapsto f_*(X) \end{aligned}$$

is a linear isomorphism, whose inverse is $(f^{-1})_*$.

As a map on derivations, the pushforward map is:

$$\begin{aligned} f_* : \{\textit{Derivations on } C^\infty(M,\mathbb{R})\} &\to \{\textit{Derivations on } C^\infty(N,\mathbb{R})\} \\ D &\mapsto D \circ f^{-1} \end{aligned}$$

Proof: Exercise (easy).

7.4 Vector fields in local coordinates

Let $\varphi = (x^1, \ldots, x^m)$ be local coordinates on $U \subseteq M$.

Any smooth vector field X can be written $X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}$ on U. The X^i's are smooth real-valued functions on U, called the ***components*** (or ***coordinates***) of X.

Remark. $X^i = \mathrm{d}x^i(X) = x^i \cdot X = \frac{\partial x^i}{\partial X}$.

Remark. $\varphi_* X = (X^1, \ldots, X^m) = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}$

Change of coordinates.

Let $(y^1, \ldots, y^m)$ be other local coordinates on $V \subseteq M$. Write $X = \sum_{i=1}^{m} Y^i \frac{\partial}{\partial y^i}$.

Proposition

Let F denote the transition function from $\varphi = (x^i)$ to $\psi = (y^j)$ on $U \cap V$.
Then $Y^j = \sum_{i=1}^{m} \frac{\partial F^j}{\partial x^i} X^i$.

Proof.

$$
\begin{aligned}
Y^j &= \mathrm{d}y^j(X) = \mathrm{d}(F^j \circ \varphi)(X) \\
&= \mathrm{d}F^j\left(\mathrm{d}\varphi(X)\right) \\
&= \mathrm{d}F^j\left(\sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}\right) \\
&= \sum_{i=1}^{m} \frac{\partial F^j}{\partial x^i} X^i .
\end{aligned}
$$

Differentials in coordinates.

Proposition

Let $(x^1, \dots, x^m)$ be local coordinates on $U \subseteq M$. Let X be a smooth vector field on U.

- *For any smooth function $f\colon U \longrightarrow \mathbb{R}$,*

$$X \cdot f = \mathrm{d}f(X) = \sum_{i=1}^{m} X^i \frac{\partial f}{\partial x^i}$$

- *For any smooth map $f\colon M \longrightarrow N$,*

$$\mathrm{d}f(X) = \sum_{i=1}^{m} X^i \,\mathrm{d}f\left(\frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} X^i \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

- *For any diffeomorphism $f\colon M \longrightarrow N$,*

$$(f_* X)|_{f(p)} = \sum_{i=1}^{m} \sum_{j=1}^{n} (X^i)|_p \frac{\partial f^j}{\partial x^i}|_p \frac{\partial}{\partial y^j}|_{f(p)}$$

Chapter 8 Flows and Lie Bracket

- 8.1 Integral curves
- 8.2 Flow of a vector field
- 8.3 Lie bracket
- 8.4 Lie derivative
- 8.5 Lie algebras
- 8.6 The Frobenius theorem

8.1 Integral curves

Let M be a smooth manifold and X a smooth vector field on M.

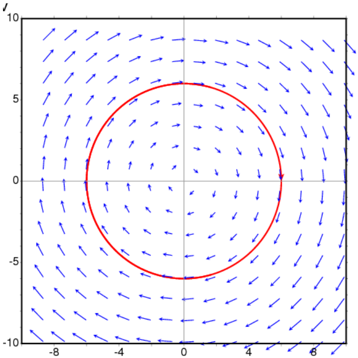
Definition

A smooth curve $\gamma : I \subseteq\longrightarrow M$ is called an ***integral curve*** of X if, for all $t \in I$:

$$\gamma'(t) = X_{\gamma(t)} \; .$$

Example 1. Consider the constant vector field $Y = \frac{\partial}{\partial y}$ in $\mathbb{R}^2$.
$\gamma(t) = (x(t), y(t))$ is an integral curve $\Leftrightarrow (x'(t), y'(t)) = (0, 1)$.
Integral curves: $\gamma(t) = (x_0, t + y_0)$. Integral curves = vertical lines.

Example 2. Consider the vector field $X = -y\frac{\partial}{\partial y} + x\frac{\partial}{\partial y}$.
Integral curves = Circles centered at the origin. Exercise.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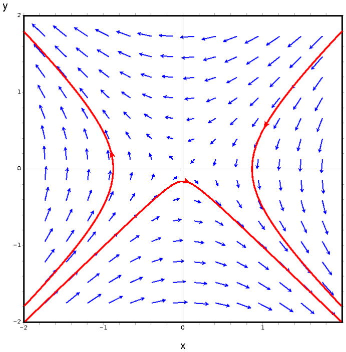
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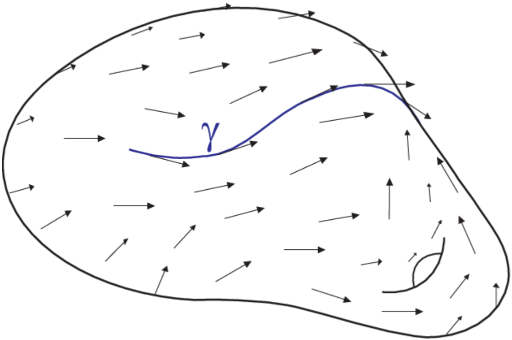

γ

Theorem

There exists a unique integral curve through any point.

More precisely: $\forall p \in M$, $\exists!$ *integral curve* $\gamma: I \to M$, *with* I *maximal, s.t.* $\gamma(0) = p$.

Proof. If $U \subseteq \mathbb{R}^m$, X is a given by $F: U \to \mathbb{R}^m$, and the equation of an integral curve is $\gamma'(t) = F(\gamma(t))$. Conclude by the Picard-Lindelöf (i.e. Cauchy-Lipschitz) theorem.

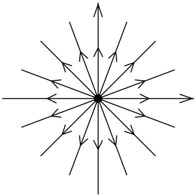
In general, use charts to apply with previous result in local charts. This shows local existence and uniqueness, and conclude.

Remark. Prerequisite: Basic theory of ODEs. Reference: [Lee, Appendix D].
In addition to the existence and uniqueness result, we have:

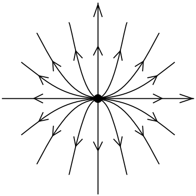
- Smooth dependence of solutions of ODEs on initial condition.
- If the maximal interval I has a finite bound, for instance $I = (a, b)$ with $b < +\infty$, then $\gamma(t)$ leaves every compact set when $t \to b^-$.

  In particular, if M is compact, then $I = \mathbb{R}$: all integral curves are ***complete***, i.e. X is complete.

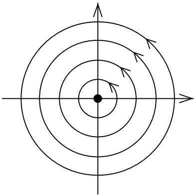
Remark. If $X|_p = 0$, then the constant curve $\gamma(t) = p$ is an integral curve.
p is called a ***zero*** or a ***singular point*** of the vector field.



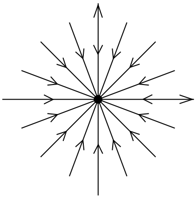
$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$



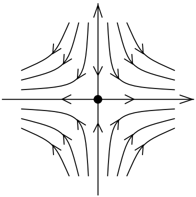
$$x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$



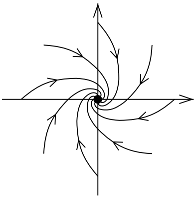
$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$



$$-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$



$$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$



$$(y-x)\frac{\partial}{\partial x} - (y+x)\frac{\partial}{\partial y}$$

8.2 Flow of a vector field

Let M be a smooth manifold and X a smooth vector field on M.

For any $p \in M$, denote $\varphi_t^X(p) := \gamma(t)$, where γ is the integral curve of X through p.

Remark. A priori, $\varphi_t^X(p)$ is only well-defined for t sufficiently small.

Theorem

- *The map* $\mathbb{R} \times M \to M$, $(t, p) \mapsto \varphi_t^X(p)$ *is smooth on its domain of definition (which is a neighborhood of* $\{0\} \times M$*).*
- $\varphi_s^X \circ \varphi_t^X(p) = \varphi_{t+s}^X(p)$ *whenever well-defined.*

Any map $\mathbb{R} \times M \to M$ as in the theorem is called a smooth ***flow*** on M.

Proof.

- Smooth dependence of solution of an ODE on initial condition.
- If γ is the integral curve through p, then so is $\gamma(t_0 + t)$ is the integral curve through $\gamma(t_0)$.

Corollary

- *$\varphi_0^X : M \longrightarrow M$ is the identity map.*
- *If φ_t^X is well-defined, then it is a diffeomorphism of M with inverse φ_{-t}^X.*
- *If well-defined, the map $t \mapsto \varphi_t^X$ is a group homomorphism $\mathbb{R} \longrightarrow \mathrm{Diff}(M)$.*

Terminology. The flow is called ***complete*** if it is defined on $\mathbb{R} \times M$, i.e. all integral curves are complete (defined on $\mathbb{R}$), i.e. X is a ***complete vector field***.

Fact. If M is compact, any vector field on M is complete.

Example. Let $X = \frac{\partial}{\partial y}$ on $M = \mathbb{R}^2$. Then $\varphi_t^X(x, y) = (x, y + t)$.

Exercise. Let $M = \mathbb{R}^2 - \{0\}$. Find two vector fields whose integral curves are rays emanating from the origin, one complete, the other incomplete.

A normal form theorem:

Theorem

Let X be a smooth vector field on M. If $p \in M$ is regular (i.e. nonsingular) point, then there exists local coordinates $x^1, \ldots, x^p$ near p s.t. $X = \frac{\partial}{\partial x^1}$.

Remark. For the proof of this theorem and more details on flows, refer to Lee's book.

8.3 The Lie bracket

Recall that a *derivation* on an algebra A is a $\mathbb{R}$-linear map $D\colon A \longrightarrow A$ s.t.

$$D(fg) = D(f)\, g + f\, D(g)$$

Fact. If D_1 and D_2 are derivations, then $D := D_1 \circ D_2 - D_2 \circ D_1$ is a derivation.

Proof. Stupid algebra computation. Do it!!

Definition

D is denoted $[D_1, D_2]$ and called the ***Lie bracket*** (or *commutator*) of D_1 and D_2.

Recall that there is a bijection between smooth vector fields and derivations on a smooth manifold, more precisely there is a linear isomorphism

$$\begin{aligned} \Gamma(\mathrm{T}M) &\longrightarrow \{\text{Derivations on } C^\infty(M,\mathbb{R})\} \\ X &\mapsto (f \mapsto X \cdot f) \end{aligned}$$

With this correspondence, we get the ***Lie bracket*** of vector fields:

Proposition

If X and Y are smooth vector fields on M, there exists a unique smooth vector field $[X,Y]$ such that for any smooth function $f \colon M \to \mathbb{R}$,

$$[X,Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot g)\,.$$

Proposition (Properties of the Lie bracket)

- $[\cdot,\cdot]$ *is* $\mathbb{R}$*-bilinear:* $[\lambda X_1 + \mu X_2, Y] = \dots$ *and* $[X, \lambda Y_1 + \mu Y_2] = \dots$
- $[\cdot,\cdot]$ *is antisymmetric:* $[Y,X] = -[X,Y]$*. In part.* $[X,X] = 0$*.*
- *Jacobi identity:* $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$*.*

Proof. Idiotic algebra computations. Do it!

Definition

A vector space A equipped with a bilinear map $[\cdot,\cdot] : A \times A \to A$ satisfying the properties above is called a ***Lie algebra***.

Examples.

1. $[\cdot,\cdot] = 0$. *(abelian Lie algebra)*
2. $\{$Derivations on $C^\infty(M,\mathbb{R})\}$
3. $\Gamma(\mathrm{T}M)$
4. Lie algebra of a Lie group (see later).

Proposition (Further properties of the Lie bracket)

- $[fX, Y] = f[X, Y] - (Y \cdot f)X$.
- *Naturality of the Lie bracket:* $f_*[X, Y] = [f_*X, f_*Y]$.

Proof. Moronic algebra computations. Do it!

Proposition (Lie bracket in coordinates)

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

$$\left[\sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}, \sum_{j=1}^{m} Y^j \frac{\partial}{\partial x^j}\right] = \sum_{i,j=1}^{m} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$$