Chapter 3 Examples of manifolds

3.1 Trivial examples

- Empty manifold
- 0-manifolds: any (countable) set with the discrete topology.
- $\mathbb{R}^m$ and open sets in $\mathbb{R}^m$.
- Finite dim. real and complex vector spaces, and their open sets.

3.2 Diffeomorphic structures

Example. Let $M = \mathbb{R}$. Take any homeomorphism $f : \mathbb{R} \to \mathbb{R}$ that is not a diffeomorphism. For instance, $f(x) = x^3$.

Then $\{(\mathbb{R}, f)\}$ is a smooth atlas on M. New smooth structure on $\mathbb{R}$.

However, the map $f : M \to M$ is a diffeomorphism from the old smooth structure to the new.

General case. Let M be a topological manifold. The group $\mathrm{Homeo}(M)$ acts on the set of smooth atlases on M by pullback.

$\to$ action of $\mathrm{Homeo}(M)$ on the set of smooth structures.

However, each orbit consists of diffeomorphic smooth structures.

3.3 Spheres

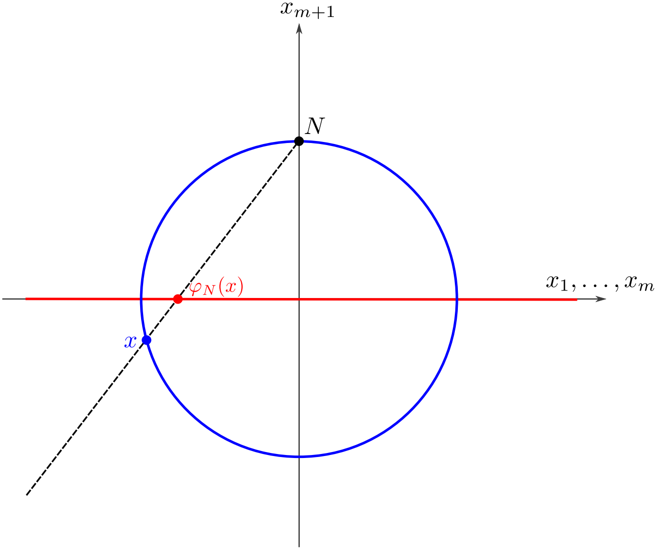
$S^m = \{x \in \mathbb{R}^{m+1} \mid \|x\| = 1\}$

Exercise. Draw or describe S^0, S^1, S^2, S^3.

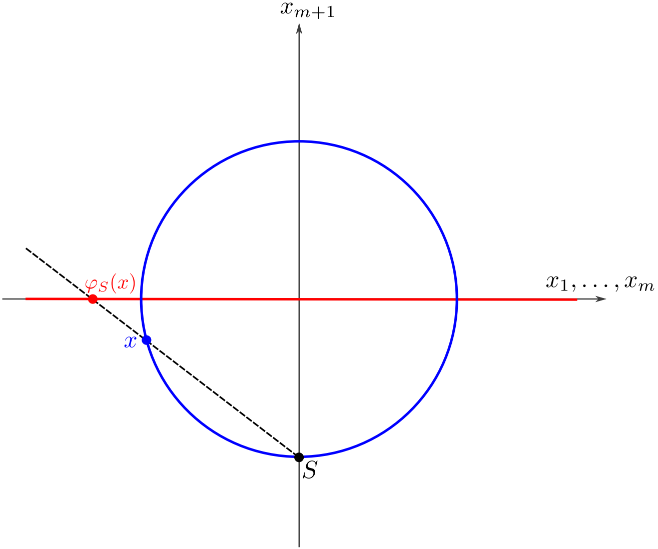
Examples of charts.

- Stereographic projection
- Spherical coordinates (see Exercise H2).
- Cartesian coordinates (locally, take all except one).
  Example: $x^2 + y^2 + z^2 = 1$. Locally, x and y determine z.

Stereographic projection.



Stereographic projection.



Stereographic projection.

Analytic expressions:

$$\varphi_N \colon S^m - \{N\} \to \mathbb{R}^m$$
$$x = (x_1, \ldots, x_{m+1}) \mapsto \frac{(x_1, \ldots, x_m)}{1 - x_{m+1}}$$

$$\varphi_S \colon S^m - \{S\} \to \mathbb{R}^m$$
$$x = (x_1, \ldots, x_{m+1}) \mapsto \frac{(x_1, \ldots, x_m)}{1 + x_{m+1}}$$

$$\varphi_S \circ \varphi_N^{-1} \colon \mathbb{R}^m - \{0\} \to \mathbb{R}^m - \{0\}$$
$$y \mapsto \frac{y}{||y||^2}$$

Some useful exercises:

Exercise. Exercise sheet #2, Exercise H2.

Exercise. Show that S^m is homeomorphic to the one-point compactification of $\mathbb{R}^m$.
Is the one-point compactification of an open manifold always a closed manifold?

Exercise. Let $S(a, r)$ be the sphere of center $a \in \mathbb{R}^{m+1}$ and center $r > 0$. Show that $S(a, r)$ is diffeomorphic to S^m.

Exercise. Let B^m be the unit ball in $\mathbb{R}^m$. Show that B^m is a smooth manifold with boundary and $\partial B^m = S^{m-1}$.

3.4 Projective spaces

Definition

Let V be a real vector space. The ***projective space of*** V is the quotient $\mathbf{P}(V) := (V - \{0\})/\sim$, where $u \sim v \Leftrightarrow u = \lambda v$ for some $\lambda \in \mathbb{R}$.

Remark: Alternatively, $\mathbf{P}(V)$ *is the set of vector lines in* V*.*

Notation: The projective space of $\mathbb{R}^{m+1}$ is denoted $\mathbb{R}P^m$.

Fact: As a topological space, $\mathbb{R}P^m \approx S^m/\sim$, where $x \sim -x$.
In part. $\mathbb{R}P^m$ is compact Hausdorff.

Proposition

$\mathbb{R}P^m$ is a closed smooth manifold of dimension m.

Proof 1: Use *affine charts* (See Exercise H1).

Proof 2: Use the action of $\{\pm 1\}$ on S^m.

Exercise. Define the complex projective space $\mathbb{C}P^m$. Show that it is a closed complex manifold.

3.5 Submanifolds

We will discuss these in Chapter 6.

Many examples!

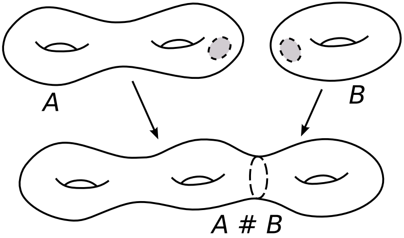
- Open subsets of manifolds.
- Boundary and interior of manifolds with boundary.
- Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, surfaces in $\mathbb{R}^3$.
- Graphs of smooth functions. Example: $z = x^2 + y^2$.
- Submanifolds defined by equations. Example: conics in $\mathbb{R}^2$, spheres $x^2 + y^2 + z^2 = 1$, affine varieties, …
- Leaves of a foliation. *Example:* $\mathbb{R}^2$ *is foliated by lines.*

3.6 Gluing constructions

Gluing manifolds: $M = \left(\bigcup_i M_i\right) /_{\sim}$

The result has no reason to be a manifold! But it can be under suitable assumptions.

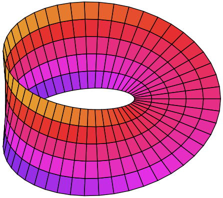
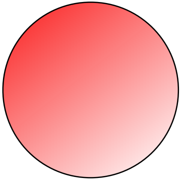
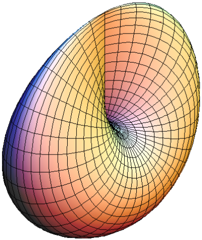
Example: Connected sum.



3.6 Gluing constructions

More generally: Gluing two manifolds with boundary along a “common” boundary component.

Example: Möbius strip + Closed disk = Projective plane $\mathbb{R}P^2$

3.7 Product manifolds

Fact. If M_1 (resp. M_2) is a smooth manifold of dim. m_1 (resp. m_2), then $M_1 \times M_2$ is a smooth manifold of dimension $m_1 + m_2$.

Examples.

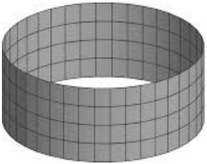
- Euclidean spaces: $R^m = \mathbb{R} \times \mathbb{R}^{m-1} = \mathbb{R} \times \cdots \times \mathbb{R}$.
- Cylinder: $S^1 \times \mathbb{R}$.
- Tori: $T^2 = S^1 \times S^1$. More generally $T^m = S^1 \times \cdots \times S^1$.

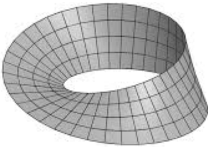
Generalization: A *fiber bundle* is a manifold that is locally a product.

More precisely: A manifold M is called a ***fiber bundle*** with base B and typical fiber F if it is equipped with a surjective map $\pi\colon M \to B$ such that for any $x \in M$, $\exists U \ni x$ and $\exists \varphi\colon \pi^{-1}(U) \xrightarrow{\sim} U \times F$ and:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\varphi\quad} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

Examples.





Example: The *Hopf fibration*.

Let

$$S^1 = \{z \in \mathbb{C} \ | \ |z| = 1\}$$
$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \ | \ |z_1|^2 + |z_2|^2 = 1\}$$

Consider the action of S^1 on S^3 by mult.: $z \cdot (z_1, z_2) = (zz_1, zz_2)$.

Exercise. The quotient S^3/S^1 is $\mathbb{C}P^1 \approx S^2$.

Proposition. The projection map $\pi \colon S^3 \to S^3/S^1 \approx S^2$ is a fiber bundle with typical fiber S^1, called the ***Hopf fibration***.

Proof. The action of S^1 on S^3 is free and proper (immediate by compactness). It is a general fact that the quotient is a manifold and the projection map is a fiber bundle.

3.8 Quotient manifolds

- Gluings: see before.
  Examples: connected sums, Möbius strip $M = [0,1] \times [0,1]/\sim$
- Quotient manifolds by the proper action of a discrete group.
  Examples: tori $T^m = \mathbb{R}^m/\mathbb{Z}^m$, Klein bottle
- Quotient manifolds by the proper action of a group.
  Example: Hopf fibration.

Theorem

Let G be a group acting on a manifold M by diffeos. If the action is free and properly discontinuous, then the quotient M/G is a manifold, and the projection $\pi : M \to M/G$ is a local diffeo.

Remark: The projection map π is actually a *covering map*, i.e. a fiber bundle with discrete typical fiber.

Proof: Adapt the proof of the topological version.

3.9 Lie groups

Definition

A ***Lie group*** G is a smooth manifold and a group such that the multiplication $G \times G \to G$ and the inversion $G \to G$ are smooth maps.

Examples:

- $(\mathbb{R}, +)$, more generally $(\mathbb{R}^m, +)$ or $(\mathbb{C}^m, +)$
- $S^1 \subseteq \mathbb{C}$ with complex multiplication.
- $S^3 \subseteq \mathbb{H}$ with quaternionic multiplication.
- $\mathbb{R}/\mathbb{Z}$ is a Lie group isomorphic to S^1.
- More generally, $T^m = \mathbb{R}^m/\mathbb{Z}^m$ is a Lie group.
- $\mathrm{GL}(n, \mathbb{R})$.

3.9 Lie groups

Definition

A ***a matrix Lie group*** G is a Lie subgroup (i.e. a submanifold and a subgroup) of $\mathrm{GL}(n, \mathbb{R})$ for some $n \in \mathbb{N}$.

Examples:

- $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n, \mathbb{R})$
- $\mathrm{GL}(n, \mathbb{C}) \subseteq \mathrm{GL}(2n, \mathbb{R})$, $\mathrm{U}(n, \mathbb{R})$
- Many othes: $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{O}(p, q)$, exceptional Lie groups, ...

Exercises.

- Show that $\mathrm{U}(1)$ is a matrix Lie group isomorphic to S^1.
- Show that $\mathrm{SU}(2)$ is a matrix Lie group isomorphic to S^3.
- Show that $\mathbb{R}$ is a matrix Lie group.

3.10 One-dimensional manifolds

Theorem

Any connected manifold of dim. 1 is diffeomorphic to $\mathbb{R}$ or S^1.

Proof.

By Whitney's theorem (smooth version), we can assume $M \subseteq \mathbb{R}^n$.
Let $\gamma\colon I \to M$ be a local diffeo where $I \subseteq \mathbb{R}$ is an open interval.

- γ exists and we can assume I maximal.
- WLOG, we can assume that γ is an arclength parametrization as a curve in $\mathbb{R}^n$.
- The image of γ is open and closed in M. Therefore, γ is surjective.
- If γ is injective, it is a diffeomorphism $I \approx M$, so we win.
- If γ is not injective, then $I = \mathbb{R}$ and γ is periodic.
  It follows that $M \approx \mathbb{R}/T\mathbb{Z} \approx S^1$.

□

3.11 Two-dimensional manifolds

Definition

A ***surface*** is a closed manifold of dimension 2.

Theorem

Any closed surface is diffeomorphic to either:

- S^2
- $T^2 \# \ldots \# T^2$
- $\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$

Corollary. A closed simply-connected surface is diffeomorphic to S^2.

Theorem. (weak version of uniformization theorem)
Any surface admits S a geometry (spherical, Euclidean, or hyperbolic).
More precisely: There exists a group G acting properly discontinuously and freely on $X = S^2$, $X = \mathbb{R}^2$, or $X = \mathbb{H}^2$ such that $S \approx X/G$.

3.12 Three-dimensional manifolds

Examples:

- $\mathbb{R}^3$, S^3, T^3, $\mathbb{R}P^3$
- $S \times S^1$, $S \times \mathbb{R}$ where S is a surface.
- Mapping torus: Let S be a closed surface and let $f \in \mathrm{Diffeo}(S)$.
  Put $M = S \times [0, 1]/\sim$ where $(x, 0) \sim (f(x), 1)$.

Poincaré conjecture. (proved by Perelman, ~2003)
Any closed simply-connected 3-manifold is diffeomorphic to S^3.

Geometrization. (conjectured by Thurston, now proven).
Every 3-manifold can be geometrized.