Chapter 2 Differential manifolds

- 2.1 Prerequisite: Differential calculus
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- 2.3 Differential structures
- 2.4 Local coordinates
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2.1 Prerequisite: Differential calculus

- Differential of a function, partial derivatives, Jacobian, gradient, chain rule.
- Functions of class C^k, Taylor expansions, real-analytic functions.
- Local inversion theorem, implicit function theorem.
- Critical points, Hessian, local extrema.
- (Multiple integrals.)

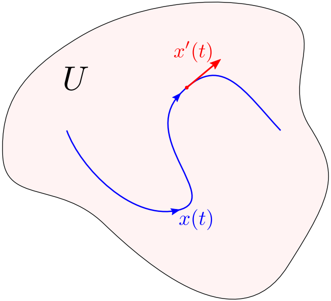
References:

- *Lee, Smooth Manifolds, Appendix C*
- *Lafontaine, Differential manifolds, Chapter 1*

Differential of a function and velocities of curves

Let $t \in I \mapsto x(t)$ be a differentiable curve in $U \subseteq \mathbb{R}^m$.

For all $t \in I$, the derivative $x'(t) \in \mathbb{R}^m$ can be thought as a vector "based at $x(t)$", called the ***velocity***.



Differential of a function and velocities of curves

Let $f\colon U \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a differentiable function.

At any $x \in U$, the differential (or derivative) of f is a linear map

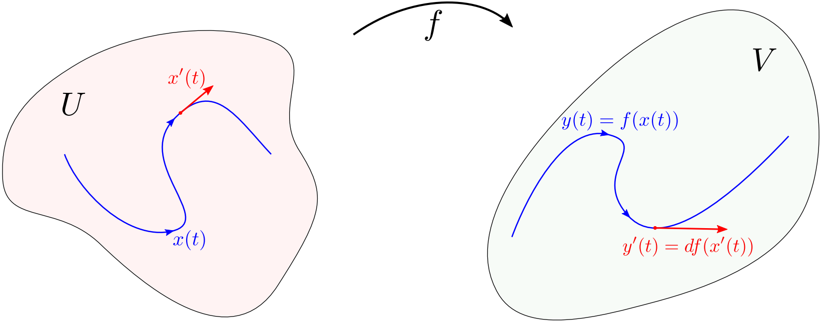
$$\mathrm{d}f_{|_x} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

Given a differentiable curve $t \mapsto x(t)$ in U, consider the image curve $t \mapsto y(t) = f(x(t))$ in $\mathbb{R}^n$.

Proposition

$t \mapsto y(t)$ is a differentiable curve in V, with velocity $y'(t) = \mathrm{d}f_{|_{x(t)}}(x'(t))$.

Differential of a function and velocities of curves



2.2 Charts and atlases

Fix a “class of regularity” C for maps between open sets of $\mathbb{R}^m$, such as:

- Continuous
- C^1
- C^k
- C^∞ (***smooth***)

Remark

Other ideas: Differentiable, PL (piecewise linear), C^ω (real-analytic), Hol (when m is even). Further: Banach, algebraic, (X, G)-structures, …(any pseudogroup)

Let M be a topological manifold of dimension m.

Recall that a ***chart*** is a pair (U, φ) where $\varphi\colon U \subseteq M \to V \subseteq \mathbb{R}^m$ is a homeomorphism.

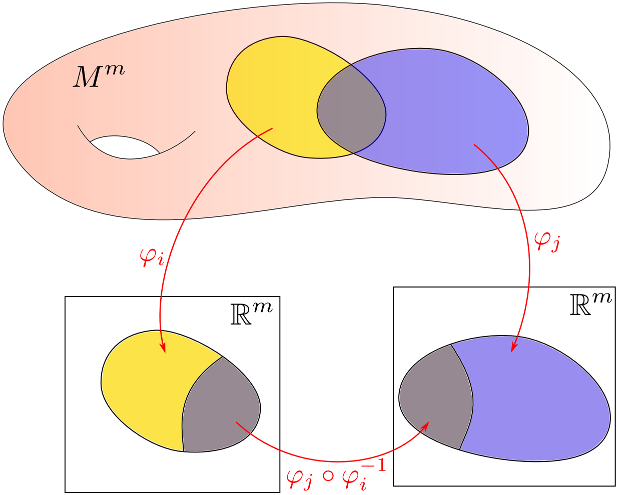
An ***atlas*** is a collection of charts $(U_i, \varphi_i)_{i \in I}$ such that $M = \bigcup_{i \in I} U_i$.

Definition

Two charts (U_1, φ_1) and (U_2, φ_2) are ***($\mathcal{C}$-)compatible*** if the transition function $\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is of class $\mathcal{C}$.

A ***$\mathcal{C}$-atlas*** is a atlas whose charts are pairwise $\mathcal{C}$-compatible.

Charts and atlases



2.3 Differential structures

Definition

Two C-atlases are called ***C-compatible*** if any chart of one is compatible with any chart of the other.

A ***C-structure*** on M is an equivalence class of compatible C-atlases.
A ***C-manifold*** is a manifold equipped with a C-structure.

Remark

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible C-atlases $\Leftrightarrow$ $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C-atlas.
- C-structure $\Leftrightarrow$ maximal C-atlas.

For example, a ***smooth manifold*** is a manifold equipped with a C^∞ structure. In practice, it is sufficient to specify *one* smoth atlas.

A ***differential manifold*** is a manifold equipped with a C-structure, where C is some regularity class contained in differentiable.

Differentiable maps

Let $f : M \to N$ be a map betwen two C-manifolds.

Definition

f is ***of class*** C if, for any compatible charts (U, φ) on M and (V, ψ) on N, the map $\psi \circ f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is of class C.

f is a C***-diffeomorphism*** if it is bijective, and f and f^{-1} are of class C.

Example

The map

$$\begin{aligned} \mathbb{R}/\mathbb{Z} &\to S^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

is a smooth diffeomorphism. Good exercise!

Remark

Existence and uniqueness of differential structures on manifolds?

Negative results:

- Some manifolds do not admit a differential structure (Kervaire 1960, E_8 manifold, ...).
- A differential structure on a manifold M is not unique, due to the action of $\mathrm{Homeo}(M)$. See Homework exercise.
  But is it unique up to diffeomorphism?
- Some manifolds admit several non-diffeomorphic differential structures: exotic spheres (Milnor), $\mathbb{R}^4$, ...

Remark

Existence and uniqueness of differential structures on manifolds?

Positive results:

- For $m \neq 4$, $\mathbb{R}^m$ admits a unique differential structure.
- Low dimensions: For $m \leqslant 3$, manifolds of dim. m admit a unique differential structure.
- Equivalence of C^k categories for all $k \in [1, +\infty]$ (ref: Hirsch).

Henceforth, we will always work in the smooth category: $C = C^\infty$.

Case $N = \mathbb{R}$: A smooth map $f : M \to \mathbb{R}$ is called a ***smooth function***.

Remark: The space $C^\infty(M, \mathbb{R})$ is an algebra.

Idea: Instead of working with the manifold M itself, work with $C^\infty(M, \mathbb{R})$ instead. (Reminiscent of E and E^* in linear algebra.)

More generally, let C be a category as before. The ***structure sheaf*** of M is $U \mapsto C(U, \mathbb{R})$. This is an example of ***locally ringed space***.

This point of view is most successful in algebraic geometry. A ***scheme*** (locally ringed space locally isomorphic to ring spectra) = best definition of an algebraic variety.

2.4 Local coordinates

Let M be a smooth manifold and let (U, φ) be a smooth chart. (aka "coordinate chart").

Denote $\varphi = (x_1, \ldots, x_m)$ the components (aka coordinates) of φ.
Each x_i is a smooth map $x_i \colon U \to \mathbb{R}$.

Definition

The m-tuple of smooth functions $(x_1, \ldots, x_m)$ is called a ***system of local coordinates*** on M.

Remark (Upper indices notation)

Ricci calculus is a set of notational conventions in differential geometry (including *Einstein notation*).
The indices in a system of coordinates should be written in superscript: $(x^1, \ldots, x^m)$ instead of $(x_1, \ldots, x_m)$.

Example (Cartesian coordinates)

On $M = \mathbb{R}^m$, we have local coordinates $(x_1, \ldots, x_m)$. Here, we abusively denote x_k the map $\mathbb{R}^m \longrightarrow \mathbb{R}$ defined by $x \mapsto x_k$.

Example (Polar coordinates)

On $\mathbb{R}^2 - \{0\}$, we have polar coordinates (ρ, θ). These are only *local* coordinates, because the map $\theta \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is not well-defined globally.

Change of coordinates

Let $(x_1, \ldots, x^m)$ and $(y_1, \ldots, y^m)$ be two systems of local coordinates. On the overlap of the coordinate charts:

$$\psi = F \circ \varphi$$

where F is the transition function. In coordinates:

$$y_k = F_k(x_1, \ldots, x_m) \, .$$

Example (Change of coordinates: from polar to Cartesian)

On $\mathbb{R}^2 - \{0\}$, let (x, y) be the Cartesian coordinates and (ρ, θ) the polar coordinates.

To be precise, let us take $\theta\colon \mathbb{R}^2 \to (-\pi, \pi]$ the principal argument function. (ρ, θ) is a system of local coordinates on $\mathbb{R}^2 - (-\infty, 0] \times \{0\}$.

The change of coordinates from polar to Cartesian is written $(x, y) = F(\rho, \theta)$ where F is the transition function from the polar chart to the Cartesian chart.

In coordinates:

$$x = \rho \cos \theta$$
$$y = \rho \sin \theta \ .$$

Other examples: Exercises.

2.5 Smooth manifolds with boundary

Let $H^m \subseteq \mathbb{R}^m$ denote the upper half-space. A map $F\colon H^m \to \mathbb{R}^n$ is *smooth* if it extends as a smooth function $U \supseteq H^m \to \mathbb{R}^n$.

Definition

A ***smooth manifold with boundary*** is a topological manifold with boundary equipped with a (equivalence class of) atlas whose transition functions are smooth maps between open sets of H^m.

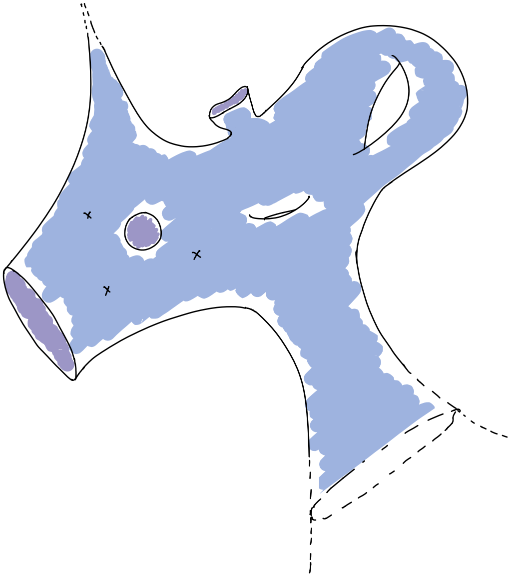
- $p \in M$ is an ***interior point*** if $\varphi(p)$ is an interior point of H^m for some/any chart φ.
- $p \in M$ is a ***boundary point*** ...is a boundary point of H^m ...
- ***Boundary of M:*** Set of boundary points, denoted ∂M.
  Exercise: ∂M (if nonempty) is a smooth $(m-1)$-manifold.
- ***Interior of M:*** Set of interior points, denoted $M - \partial M$.
  Exercise: $M - \partial M$ is a smooth m-manifold.

Examples

- See Chapter 1.
- M = *Any smooth manifold* $\quad \partial M = \emptyset$.
- $M = [-1, 1) \cup [2, +\infty) \quad \partial M = \{-1\} \cup \{2\}$.
- $M = B^m \subseteq \mathbb{R}^m \quad \partial M = S^{m-1}$. *Exercise!*
- $M = S^1 \times [0, 1] \quad \partial M = S^1 \times \{0\} \cup S^1 \times \{1\}$.

Smooth manifolds with boundary

$M =$



Smooth manifolds with boundary

$\partial M =$