Chapter 12 Integration and Stokes's theorem

- 12.1 Preamble: Integration of differential forms in $\mathbb{R}^m$.
- 12.2 Integration on manifolds
- 12.3 Stokes's theorem

12.1 Preamble: Integration of differential forms in $\mathbb{R}^m$.

Prerequisites:

- Multiple integrals. Change of variables formula, etc.

  $\rightarrow$ *Very brief recap in [Lee, Smooth Manifolds] in Appendix C.*
  $\rightarrow$ *For more in-depth review, go to your favorite textbook.*
  If you don't have one, try: [Duistermaat, Kolk: Multidimensional Real Analysis II]

- Measure theory and Lebesgue integral: useful, but not necessary.

  $\rightarrow$ *If you want to review this, use your favorite textbook.*
  If you don't have one, try:
    - *[Stein, Shakarchi: Real analysis]*
    - *[Jones, Lebesgue integration on Euclidean space]*

Let $U \subseteq \mathbb{R}^m$ and $f\colon U \to \mathbb{R}$. One can define $\int_U f$.

Notations:

$$\int_U f(x_1, \ldots, x_m)\,\mathrm{d}x_1 \ldots \mathrm{d}x_m \qquad\qquad \int_U f(x)\,\mathrm{d}\lambda(x)$$

Remark. f needs to be integrable! This is not something we will worry about, by imposing the comfortable restrictions:

1. $f \in C^\infty(U, \mathbb{R})$.
2. Either U is bounded and $f \in C^\infty(\bar{U}, \mathbb{R})$,
   or f has compact support in U. Recall: $\operatorname{Supp} f = \overline{\{f \neq 0\}}$.

Let ω be a differential form of top degree on U: $\omega \in \Omega^m(U, \mathbb{R})$.

On $U \subseteq \mathbb{R}^m$ we have canonical coordinates $(x^1, \ldots, x^m)$, and ω can be written

$$\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

where $f \in C^\infty(U, \mathbb{R})$.

Remark. At every point $x \in U \subseteq \mathbb{R}^m$, the vector space $\Lambda^m \mathrm{T}^* U = \Lambda^m \mathrm{T}^* \mathbb{R}^m$ is one-dimensional. (Indeed, $\binom{m}{m} = 1$.)

The basis element $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m{}_{|x} = (e^1)^* \wedge \cdots \wedge (e^m)^*$, as an alternating multilinear form $\mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R}$, is nothing else than the determinant! (Indeed, it takes the same value on the tuple $(e_1, \ldots, e_m)$.)

Definition

The integral of $\omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ on $U \subseteq \mathbb{R}^m$ is

$$\int_U \omega := \int_U f(x)\,\mathrm{d}\lambda(x)\ .$$

Remark. In other words, we put:

$$\int_U f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m := \int_U f\,\mathrm{d}x^1 \ldots \mathrm{d}x^m\ .$$

We just erased the wedges! We will soon see the idea of this definition.

Proposition

Let $F\colon U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^m$ be an orientation-preserving diffeomorphism.
For any $\omega \in \Omega^m(V, \mathbb{R})$, we have:

$$\int_U F^*\omega = \int_V \omega \, .$$

If F is orientation-reversing, then

$$\int_U F^*\omega = -\int_V \omega \, .$$

A few preliminaries before the proof:

- The ***Jacobian determinant*** of F at $x \in U$ is $J_x(f) := \det(\mathrm{d}F_x)$.
  (In other words, it is the determinant of the Jacobian matrix.)
- F is ***orientation-preserving*** [***reversing***] if $J_x(f) > 0$ [< 0] for all $x \in U$.
  Remark: If U is connected, F must be either orientation-preserving or reversing.

Lemma. For any $L \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^m)$ and for any $u_1, \ldots, u_m \in \mathbb{R}^m$, we have

$$\det(Lu_1, \ldots, Lu_m) = (\det L)\ \det(u_1, \ldots, u_m)\ .$$

Proof. This is essentially the definition of the determinant of a linear map.

Lemma. For any $\omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ on $V \subseteq \mathbb{R}^m$ and $F \in C^\infty(U, V)$, we have:

$$F^*\omega_{|_x} = f(F(x))\, J_x(F)\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m\ .$$

Proof. By definition of the pullback, we have for any $u_1, \ldots, u_m \in \mathbb{R}^m$:

$$\begin{aligned}
F^*\omega_{|_x}(u_1, \ldots, u_m) &= \omega_{|_{F(x)}}(\mathrm{d}F_{|_x}(u_1), \ldots, (\mathrm{d}F_{|_x}(u_m)) \\
&= f(F(x))\ (\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m)(\mathrm{d}F_{|_x}(u_1), \ldots, (\mathrm{d}F_{|_x}(u_m)) \\
&= f(F(x))\ (\det \mathrm{d}F_{|_x})\ \det(u_1, \ldots, u_m) \\
&= f(F(x))\ J_x(F)\ (\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m)_{|_x}(u_1, \ldots, u_m)\ .
\end{aligned}$$

Proposition

Let $F\colon U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^m$ be an orientation-preserving [resp. reversing] diffeo. For any $\omega \in \Omega^m(V, \mathbb{R})$, we have:

$$\int_U F^*\omega = \int_V \omega \qquad \left[\textit{resp. } \int_U F^*\omega = -\int_V \omega\right].$$

Proof. Let $\omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$. By definition, $\int_V \omega = \int_V f(x)\,\mathrm{d}\lambda(x)$.

The change of variables formula says that:

$$\int_V f(x)\,\mathrm{d}\lambda(x) = \int_U f(F(x))|J_x(F)|\,\mathrm{d}\lambda(x)\,.$$

If F is orientation-preserving, then $|J_x(F)| = J_x(F)$ so the RHS is $\int_U F^*\omega$ by the previous lemma. We're done.

Of course, if F is orientation-reversing, we have similarly that $|J_x(F)| = -J_x(F)$ so the RHS is $-\int_U F^*\omega$, we're done.

Example.

Let $I = (a, b) \subseteq \mathbb{R}$ be a bounded open interval and $f \in C^\infty([a, b])$.
We consider the 1-form $\omega := f \, \mathrm{d}x \in \Omega^1(I, \mathbb{R})$.

By definition, $\int_I \omega = \int_I f(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x$.

Now consider an orientation-preserving (i.e. increasing) diffeo $\varphi\colon J = (c, d) \to I$.
We proved that $\int_J \varphi^* \omega = \int_I \omega$. Here, $\varphi^* \omega_{|_x} = f(\varphi(x))\varphi'(x) \, \mathrm{d}x$.

In other words, we recovered the change of variables formula for one-dim integrals:

$$\int_c^d f(\varphi(x))\varphi'(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x \, .$$

Note that if φ is decreasing, we need to put a minus sign, as expected.

12.2 Integration on manifolds

Orientation.

Let us start with vector spaces.

An orientation of an m-dimensional vector space V is equivalently a choice of:

- An equivalence class of bases of V.
  $\mathcal{B} \sim \mathcal{C}$ *if* $\det P_{\mathcal{B},\mathcal{C}} > 0$ *where* $P_{\mathcal{B},\mathcal{C}}$ *is the change of basis matrix.*
- An equivalence class of bases of $\Lambda^m V^*$.
  $\dim \Lambda^m V^* = 1$*, so a basis has one element.* $\omega \sim \omega' \Leftrightarrow \omega = \lambda\omega'$ *with* $\lambda > 0$*.*

Any finite-dim real vector space has two orientations.

$f \in \mathrm{End}_{\mathbb{R}} V$ is orientation-preserving [resp. -reversing] if $\det f > 0$ [resp. < 0].

Remark. We have seen that a differentiable map $F\colon U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is called orientation-preserving if it has everywhere positive Jacobian.
i.e., F is orientation-preserving if $\mathrm{d}F_{|_x} \in \mathrm{End}(\mathbb{R}^m)$ is orientation-preserving $\forall x \in \mathrm{U}$.

Definition

Let M be a smooth manifold of dim. m.

- A smooth atlas on M is called ***an oriented atlas*** if all its transition functions are orientation-preserving.
  An orientation of M is the choice of such an atlas. More precisely:
- An ***orientation*** of M is an equivalence class of compatible oriented atlases.
- A manifold is called ***orientable*** if it admits an orientation.
  It is called ***oriented*** if it is given a choice of orientation.

Remark. Not all manifolds admit an orientation! For instance:

- The sphere S^2 and connected sums of tori $S_g = T^2 \# \dots \# T^2$ are orientable.
- Connected sums of projective planes $S_g^{\text{non}} = \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$ are non-orientable.
- The projective space $\mathbb{R}P^m$ is orientable iff m is odd. (See Exercises.)

Remark. A nonvanishing element of $\Omega^m(M, \mathbb{R})$ is called a ***volume form***.
A manifold is orientable iff it admits a volume form. (Exercise or [Lafontaine, Thm 6.5].)

Definition of the integral.

The ***support*** of a differential form $\alpha \in \Omega^k(M,\mathbb{R})$ is $\operatorname{Supp}\alpha := \overline{\{p \in M \mid \alpha_{|_p} \neq 0\}}$.

We denote $\Omega^k_c(M,\mathbb{R})$ the space of k-forms with compact support.

Theorem

Let M be an oriented m-manifold. $\exists!$ linear map $I : \Omega^m_c(M,\mathbb{R}) \to \mathbb{R}$ such that:
For any chart (U,φ) compatible with the orientation of M and for any $\omega \in \Omega^c(M,\mathbb{R})$ with compact support in U, $I(\omega) = \int_{\varphi(U)} (\varphi^{-1})^*\omega$.

Definition

For any $\omega \in \Omega^m_c(M,\mathbb{R})$, $I(\omega)$ is called the ***integral*** of ω and denoted $\int_M \omega$.

Remark. If we change the orientation of M, we change the integral by a minus sign.
This generalizes the conventional fact that $\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t$.

Proof of the theorem.

Uniqueness.

Let $(U_i, \varphi_i)_{i \in I}$ be an oriented atlas. Let $(\rho_i)_{i \in I}$ be a subordinate partition of unity.
Partitions of unity exist because M is paracompact, see Chapter 1.

Any $\omega \in \Omega_c^m(M, \mathbb{R})$ may be written $\omega = \sum_{i \in I} \omega_i$ where $\omega_i := \rho_i \omega \in \Omega_c^m(U_i, \mathbb{R})$.

By linearity, one must have $I(\omega) = \sum_{i \in I} I(\omega_i)$.
(Here we can take a sum over the U_i's intersecting the support of ω, and assume there are finitely many of these.)

By assumption, $I(\omega_i) = \int_{\varphi_i(U_i)} (\varphi^{-1})^* \omega_i$ is completely determined, hence so is $I(\omega)$.

Proof of the theorem.

Existence.

We have given a formula for $I(\omega)$ given an oriented atlas (U_i, φ_i) and a subordinate partition of unity $(\rho_i)_{i\in I}$: $I(\omega) = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \omega_i$ where $\omega_i = \omega\rho_i$.

We must show that for another choice $(V_j, \psi_j)_{j\in J}$ and $(\theta_j)_{j\in J}$, the result is the same, i.e. $J(\omega) = I(\omega)$, where $J(\omega) = \sum_j \int_{\varphi_j(V_j)} (\psi_j^{-1})^* \omega_j$ where $\omega_j = \omega\theta_j$.

Write $\omega = \sum_{I\times J} \omega_{ij}$ where $\omega_{ij} = \omega_i \rho_i \theta_j$.

Since $\omega_i = \sum_j \omega_{ij}$, we have $I(\omega) = \sum_{I\times J} \int_{\varphi_i(U_{ij})} (\varphi_i^{-1})^* \omega_{ij}$.
Similarly, $J(\omega) = \sum_{I\times J} \int_{\psi_j(U_{ij})} (\psi_j^{-1})^* \omega_{ij}$.

We conclude by arguing that $\int_{\varphi_i(U_{ij})} (\varphi_i^{-1})^* \omega_{ij} = \int_{\psi_j(U_{ij})} (\psi_j^{-1})^* \omega_{ij}$.
This is a direct application of the Proposition of 12.1 with $F = \psi_j \circ \varphi_i^{-1}$.

The case of manifolds with boundary.

Let $A \subseteq \mathbb{R}^m$ be a subset, not necessarily open.
Example: A is an open subset of the upper half-space $H^m \subseteq \mathbb{R}^m$.

Recall that by definition, a function $f\colon A \to \mathbb{R}$ is smooth if it is the restriction of a smooth function on an open set containing A.

More generally, a smooth k-form on A is the restriction of a smooth k-form on an open set $U \subseteq \mathbb{R}^m$ containing A.

Thus $\Omega^k(A, \mathbb{R})$ is well-defined. More generally, if M is a smooth manifold with boundary, $\Omega^k(M, \mathbb{R})$ is well-defined. (Use charts with values in H^m).

If $\omega \in \Omega^m(A, \mathbb{R})$, i.e. $\omega = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ with $f \in C^\infty(M, \mathbb{R})$,
we define as before $\int_A \omega := \int_A f(x)\, \mathrm{d}\lambda(x)$.

If M is an oriented manifold with boundary, using charts with values in H^m we define $\Omega^k(M, \mathbb{R})$, and the integral $I\colon \Omega^k_c(M, \mathbb{R}) \to \mathbb{R}$ as before.

Remark. If M is a smooth manifold with boundary, then ∂M is a smooth manifold (without boundary). Moreover, $\iota\colon \partial M \to M$ is an embedding.
If $\omega \in \Omega^k(M, \mathbb{R})$, then $\omega_{|_{\partial M}} := \iota^* \omega \in \Omega^k(\partial M, \mathbb{R})$.

Further topic: Densities (off-topic).

Roughly speaking, a ***density*** on M is an object that looks like:

$$\mu = f \left| \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \right|$$

where f is a smooth function.

Densities are the “right” kind of object that can be integrated on manifolds, because their integral transforms correctly under change of coordinates.

Moreover, they have the huge advantage over m-forms that they do not care about an orientation of M, in fact integration of densities works on non-orientable manifolds.

That being said, when M is orientable, the choice of an orientation of M gives a natural identification between densities and m-forms.

To learn more: [Lee, Chap. 16].

12.3 Stokes's theorem

Theorem (Stokes's theorem)

Let M be a smooth oriented m-manifold with boundary. For any $\omega \in \Omega_c^{m-1}(M, \mathbb{R})$,

$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega .$$

Remark.

- On the left, we should write $\omega_{|\partial M}$ i.e. $\iota^* \omega$, where $\iota \colon \partial M \to M$ is the inclusion.
- An orientation of M induces an orientation of ∂M.
  Indeed, let ω be a volume form on M giving its orientation, and let N be an *outward-pointing* vector field on ∂M. Take the volume form $i_N \omega$ on ∂M.
  For more details, refer to [Lee, Chap. 15].

Example. If $M = [a, b] \subseteq \mathbb{R}$, then $\omega \in \Omega_c^0(M, \mathbb{R})$ is a function $\omega = f \in C^\infty([a, b], \mathbb{R})$.
The integral of ω on $\partial M = \{a\} \cup \{b\}$ is $f(b) - f(a)$, and $\mathrm{d}\omega$ is the 1-form $f'(t)\,\mathrm{d}t$.
Thus Stokes's theorem in this case reads $f(b) - f(a) = \int_a^b f'(t)\,\mathrm{d}t$.
In conclusion: Stokes is a generalization of the fundamental theorem of calculus.

Proof of Stokes's theorem.

Case 1: $M = U \subseteq \mathbb{R}^m$.

In this case, Stokes reduces to Fubini's theorem and the fund. theorem of calculus:

Since ω has compact support, we can assume that $\operatorname{Supp}\omega \subseteq [-R, R]^m$.

On the one hand, since $\partial M = \emptyset$, we have $\int_{\partial M} \omega = 0$. Let us prove that $\int_U \mathrm{d}\omega = 0$.

Write $\omega = \sum_{i=1}^m \omega_i \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m$. We have

$$\begin{aligned}\mathrm{d}\omega &= \sum_{i=1}^m \sum_{j=1}^m \frac{\partial \omega_i}{\partial x^j} \, \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m \\ &= \sum_{i=1}^m \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m \\ &= \sum_{i=1}^m (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \,.\end{aligned}$$

Proof of Stokes's theorem.

Case 1: $M = U \subseteq \mathbb{R}^m$.

By definition of the integral, we have

$$\int_U \mathrm{d}\omega = \sum_{i=1}^{m} (-1)^{i-1} \int_U \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \dots \mathrm{d}x^m$$
$$= \sum_{i=1}^{m} (-1)^{i-1} \int_{[-R,R]^m} \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \dots \mathrm{d}x^m \, .$$

By Fubini's theorem, we write

$$\int_{[-R,R]^m} \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^1 \dots \mathrm{d}x^m = \left(\int_{[-R,R]^{m-1}} \mathrm{d}x^1 \dots \widehat{\mathrm{d}x^i} \dots \mathrm{d}x^m \right) \left(\int_{-R}^{R} \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^i \right) \, .$$

By the fund. theorem of calculus,

$$\int_{-R}^{R} \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^i = \omega_i(R) - \omega_i(-R) = 0 \, .$$

Proof of Stokes's theorem.

Case 2: $M = U \subseteq H^m$.

The proof is exactly the same, with a minor modification. Exercise.

Case 3: The general case.

Use an atlas and a subordinate partition of unity to reduce to the previous cases.

Application to vector calculus.

Definition

A ***regular domain*** $D \subseteq \mathbb{R}^m$ is a properly embedded m-submanifold with boundary.

Remarks.

- The definition generalizes to any smooth m-manifold instead of $\mathbb{R}^m$.
- An embedded submanifold is properly embedded iff it is a closed subset.
- One can show that $D \subseteq \mathbb{R}^m$ is a regular domain iff D is equal to the closure of its interior and its boundary is a smooth embedded submanifold.

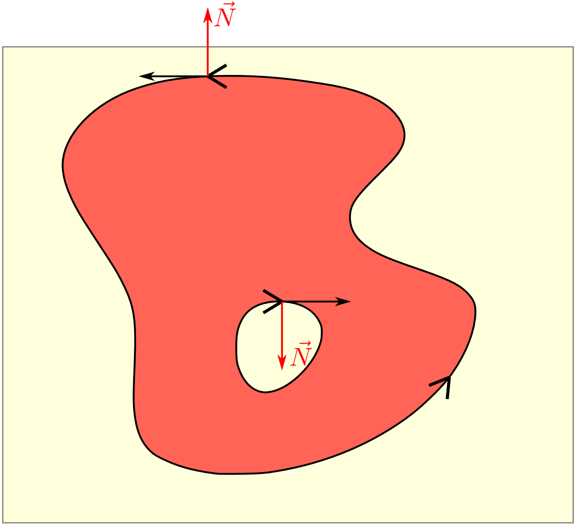

Figure: A regular domain in $\mathbb{R}^2$.

Application of Stokes in $\mathbb{R}^2$.

Theorem (Green's theorem)

Let $D \subseteq \mathbb{R}^2$ be a compact regular domain and let $P, Q\colon D \to \mathbb{R}$ be smooth functions.

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y = \int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y\,.$$

Proof. Apply Stokes's theorem to $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$.

Application of Stokes in $\mathbb{R}^3$. (Just mention, Exercises or [Lafontaine 6.4.3] for details)

Theorem (Divergence theorem)

Let $D \subseteq \mathbb{R}^3$ be a compact regular domain and let $\vec{F}$ be a smooth vector field on D.

$$\int_D \operatorname{div} \vec{F}\ \mathrm{d}V = \int_{\partial D} \langle \vec{F}, \vec{N} \rangle\,\mathrm{d}\sigma$$

where $\vec{N}$ is the outward-pointing unit normal.

Further topic: Integration on chains and singular homology (off-topic).

Let M be a compact m-manifold with boundary.

We have defined the integral of m-forms on M and $(m-1)$-forms on ∂M, and we have the Stokes theorem $\int_{\partial M} \omega = \int_M \mathrm{d}\omega$ for all $\omega \in \Omega^{m-1}(M, \mathbb{R})$.

More generally, if $A \subseteq M$ is a k-submanifold with boundary and $\omega \in \Omega^{k-1}(M, \mathbb{R})$, we have $\int_{\partial A} \omega = \int_A \mathrm{d}\omega$ ("Stokes theorem for submanifolds").

Let $C_k(M, \mathbb{R})$ denote the vector space of formal linear combinations of k-submanifolds of M with boundary. Extend the boundary map linearly $\partial\colon C_k(M, \mathbb{R}) \to C_{k-1}(M, \mathbb{R})$. Notice that $\partial \circ \partial = 0$ (the boundary of a manifold with boundary has no boundary).

The Stokes theorem for submanifold immediately extends to chains:
$\int_{\partial A} \omega = \int_A \mathrm{d}\omega$ for all $A \in C^k(M, \mathbb{R})$ and $\omega \in \Omega^{k-1}(M, \mathbb{R})$.

Integration on chains and singular homology (off-topic).

We have an obvious "duality pairing" between $\Omega^k(M,\mathbb{R})$ and $C_k(M,\mathbb{R})$:

$$(\omega, A) \mapsto \int_A \omega$$

The Stokes theorem for chains $\int_{\partial A} \omega = \int_A \mathrm{d}\omega$ says that $\partial\colon C_k(M,\mathbb{R}) \to C_{k-1}(M,\mathbb{R})$ and $\mathrm{d}\colon \Omega^{k-1}(M,\mathbb{R}) \to \Omega^k(M,\mathbb{R})$ are (formally) the dual of each other.

i.e., integration is a duality between the de Rham complex and the singular complex:

$$0 \xrightarrow{\mathrm{d}} \Omega^0(M,\mathbb{R}) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^m(M,\mathbb{R}) \xrightarrow{\mathrm{d}} 0$$
$$0 \xleftarrow{\partial} C^0(M,\mathbb{R}) \xleftarrow{\partial} \dots \xleftarrow{\partial} C^m(M,\mathbb{R}) \xleftarrow{\partial} 0$$

This duality induces a duality between $H^k_{\mathrm{dR}}(M,\mathbb{R})$ and $H_k(M,\mathbb{R}) := \ker\partial/\operatorname{Im}\partial$.
This is essentially the theorem of de Rham.

The main simplification of this story is the definition of $C_k(M,\mathbb{R})$.
For the real story, refer to [Lee, Chap. 18].