# Manifolds
## Exercise Sheet 5.

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## Groupwork

**Exercise G1** (True or false?)

True or false? Justify your answers.

a) Any smooth function on $S^1$ is not injective.

b) Any smooth vector field on $S^1$ has a zero.

c) Let $M$ and $N$ be smooth manifolds of the same dimension. For any vector fields $X$ on $M$ and $Y$ on $N$, locally there always exist diffeomorphisms $\varphi \colon U \subseteq M \to V \subseteq N$ such that $\varphi_* X = Y$.

d) The flow $\varphi_t$ of a vector field is well-defined for $t > 0$ sufficiently small.

e) For any $\alpha, \beta \in \Lambda(V^*)$, $\alpha \wedge \beta = -\beta \wedge \alpha$.

**Exercise G2** (Vector fields: computations in local coordinates)

a) Let $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$. Show that $F(x, y) = (xy, y/x)$ defines a diffeomorphism of $M$. Compute $F_* X$ where $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

b) Let $M$ be and $X$ be as above. Compute $X$ in polar coordinates.

c) Let $M = \mathbb{R}^3$. Compute the Lie bracket $[X, Y]$, where $X = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial y}$.

d) Let $M = \mathbb{R}^2$. Compute the flow of $X = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$.

**Exercise G3** (Tensor products and dimension)

Let $V$ be a vector space with a basis $(e_1, \ldots, e_n)$. Let $k \in \mathbb{N}$.

*You may start by doing the whole exercise for the case $k = 2$, and if you succeed, do the general case.*

a) Find a basis of $\mathrm{T}^k(V)$. What is the dimension of $\mathrm{T}^k(V)$?

b) Same question for $\Lambda^k(V)$.

c) Same question for $\mathrm{S}^k(V)$.

d) Is it true that $\mathrm{T}^k(V) = \mathrm{S}^k(V) \oplus \Lambda^k(V)$?

## Homework

**Hand in your work by 30.06.2020.**

**Exercise H1** (True or False?) **8 points**

True or False? Carefully prove each answer.

a) Let $M$ be a smooth manifold. Locally, one can always find vector fields $X_1, \ldots, X_m$ which form a basis of the tangent space at every point.

b) For any Lie algebra $L$ and any $X \in L$, we have $[X, X] = 0$.

c) For any vector space $V$ of dimension $n$, we have $\dim \Lambda^n V^* = 1$.

d) If $\alpha$ is a symmetric tensor, then $\mathrm{Alt}(\alpha) = 0$.

**Exercise H2** (Vector fields: computations in local coordinates) **8 points**

a) Let $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$ and $F(x, y) = (xy, y/x)$ a diffeomorphism of $M$. Compute $F_* X$ where $X = y\frac{\partial}{\partial x}$.

b) Let $M$ be as above. Compute $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and also $X = (x^2 + y^2)\frac{\partial}{\partial x}$ in polar coordinates.

c) Let $M = \mathbb{R}^3$. Compute the Lie bracket $[X, Y]$, where $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$.

d) Let $M = \mathbb{R}^2$. Compute the flow of $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$.

**Exercise H3** (Lie groups and Lie algebras) **10 points**

Let $G$ be a Lie group. Denote $e \in G$ the neutral element.

a) Show that for any $g \in G$, the map $L_g \colon h \mapsto gh$ is a smooth diffeomorphism of $G$.

b) Let $L = \{X \in \Gamma(\mathrm{T}\, M) \mid \forall g \in G\ (L_g)_* X = X\}$. Show that $L$ is a Lie subalgebra of $\Gamma(\mathrm{T}\, M)$, that is a vector subspace stable under the Lie bracket.

c) Show that $X \mapsto X_{|e}$ is a linear isomorphism $L \to \mathrm{T}_e\, G$. Derive that $\mathrm{T}_e\, G$ can be equipped with the structure of a finite-dimensional Lie algebra. *This Lie algebra is called the Lie algebra of $G$.*

d) Let $G = \mathsf{GL}(n, \mathbb{R})$. Show that the Lie algebra of $G$ is $\mathrm{M}(n, \mathbb{R})$, with $[A, B] = AB - BA$.

## Further Exercises

**Exercise F1** (Hessian)

Let $f \colon M \to \mathbb{R}$ be a smooth function. Show that it is not possible to give a sensible definition of the Hessian of $f$ at an arbitrary point $p \in M$. Show that however, the Hessian of $f$ is well-defined at a critical point.

**Exercise F2** (Compact manifolds admitting non-vanishing vector fields)

Let $M$ be a smooth compact manifold that admits a nowhere vanishing vector field. Show that there exists a smooth map $F \colon M \to M$ that is homotopic to the identity and has no fixed points.

*By definition, $F$ is homotopic to the identity if there exists a smooth map $H \colon [0,1] \times M \to M$ such that $H(0, \cdot) = \mathrm{id}_M$ and $H(1, \cdot) = F$.*

**Exercise F3** (Commuting flows)

Let $X$ and $Y$ be two vector fields on a smooth manifold $M$. For comfort, let us assume $X$ and $Y$ are complete. Show that the following are equivalent:

(i) $X$ and $Y$ commute, that is: $[X, Y] = 0$.

(ii) $X$ is invariant under the flow of $Y$: $(\varphi_t^Y)_* X = X$ for all $t \in \mathbb{R}$.

(iii) $Y$ is invariant under the flow of $X$: $(\varphi_t^X)_* Y = Y$ for all $t \in \mathbb{R}$.

(iv) $X$ and $Y$ have commuting flows: $\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X$ for all $s, t \in \mathbb{R}$.

**Exercise F4** (Classical Lie algebras)

Describe the Lie algebras of all the Lie groups you can think of.