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Hyperbolic Geometry

Winter 2019-2020

Exercise Sheet 6 (Chapter 8)

Chapter 8

Exercise 1. Poincaré metric

Feel free to take $n = 2$ in this exercise. You can always do the general case afterwards.

(1) Recover the expression of the stereographic projection $s\colon \mathcal{H}^+ \longrightarrow B^n$.

(2) Recall the expressions of the Riemannian metrics $g_{\mathcal{H}^+}$ and g_{B^n} and recover the fact that s is a Riemannian isometry.

(3) Recover the expression of the Cayley transform $c\colon H^n \longrightarrow B^n$.

(4) Recall the expression of the metric g_{H^n} and recover that c is a Riemannian isometry.

Exercise 2. Curvature of the Poincaré metric

Let $\Omega \subseteq \mathbb{R}^n$ and let $g = e^{2\varphi} g_0$ be a conformal metric in Ω. Let u, v be an orthonormal pair of vectors in $\mathbb{R}^n$ and denote P the plane spanned by u and v. The following formula (reference: [**?**]) gives the sectional curvature of the metric g at a point $x \in \Omega$ in the direction of P:

$$K_P = -e^{-2\varphi}\left[\mathrm{D}^2\varphi(u,u) + \mathrm{D}^2\varphi(v,v) + \|\nabla\varphi\|^2 - \langle\nabla\varphi, u\rangle^2 - \langle\nabla\varphi, v\rangle^2\right] .$$

(We have denoted $\nabla\varphi$ the gradient of φ.)

(1) Recover the curvature of the Poincaré metric in B^n by direct computation.

(2) Let $K < 0$. Can you find a metric of constant sectional curvature K in B^n?

(3) Same questions for H^n.

Exercise 3. Poincaré vs Klein ball

(1) Show that the natural identification between the Poincaré ball and the Beltrami-Klein ball is given by the map

$$\begin{aligned} \varphi\colon B^n_{\mathrm{P}} &\longrightarrow B^n_{\mathrm{K}} \\ x &\longmapsto \frac{2x}{1 + \|x\|^2} . \end{aligned}$$

(2) Recover that φ is a Riemannian isometry by direct computation. *Feel free to take $n = 2$.*

Exercise 4. Poincaré vs Klein ball: the distance

(1) Let x, x' be two real numbers in $[0,1)$ such that $x' = \frac{2x}{1+x^2}$. Show that $\frac{1+x'}{1-x'} = \left(\frac{1+x}{1-x}\right)^2$ and derive that $\operatorname{artanh} x' = 2 \operatorname{artanh} x$.

(2) Recover the fact that the map φ of Exercise 3 is a metric isometry, i.e. $d(\varphi(x), \varphi(y)) = d(x, y)$, in the case $y = 0$.

Exercise 5. Poincaré vs Klein ball: isometries

$\mathrm{PO}(n,1)$ acts by isometries on the Klein ball and the Poincare ball. Is this the same action on B^n? Show that the map φ of Exercise 3 conjugates the two actions.

Exercise 6. Hemisphere model

Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and denote S^n_+ the upper hemisphere (with $x_{n+1} > 0$). We also denote $S = (0, \dots, 0, -1)$ the "South pole" of S^n. We recall that the Poincaré ball may be seen as the unit ball in $R^n \subseteq R^{n+1}$.

(1) Consider the stereographic projection $s \colon S^n \to \widehat{\mathbb{R}^n}$. Find its analytic expression. Show that s restricts to a diffeomorphism $S^n_+ \to B^n$.

(2) By definition, the *hemisphere model* $(S^n_+, g_{S^n_+})$ of hyperbolic space is the inverse image of the Poincaré ball (B^n, g_{B^n}) by the stereographic projection s. Prove that $g_{S^n_+}$ can be written:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2}{x_{n+1}^2}\,.$$

In what sense is the hemisphere model a conformal model?

Exercise 7. Relations between models

(1) Show that the different models of hyperbolic space are related as showed by the diagram in Figure 1.

(2) Show that geodesics in the hemisphere model are semi-circles that are orthogonal to the equator. Explain Figure 2.

(3) Recover that geodesics in the Poincaré half-space model are semi-circles that are orthogonal to the boundary.

Exercise 8. Matrix model of hyperbolic 3-space

Let H denote the set of 2×2 matrices with complex coefficients that are Hermitian symmetric:

$$H = \{A \in \mathcal{M}_{2\times 2}(\mathbb{C}) \mid A^* = A\}$$

where we denote $A^* = {}^t\bar{A}$.

(1) Let $q(A) = -\det(A)$. Show that $q(A)$ is a quadratic form on H, with associated symmetric bilinear form $b(A, B) = -\frac{1}{2}\operatorname{tr}(A\,{}^t\mathrm{Comat}(B))$.

(2) Show that (H, b) is isomorphic to $\mathbb{R}^{3,1}$ via

$$(x_1, x_2, x_3, x_4) \mapsto \begin{bmatrix} x_1 + x_4 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 - x_4 \end{bmatrix}.$$

(3) Let $H_1 = H \cap \mathrm{SL}(2, \mathbb{C})$. Show that H_1 is a model of hyperbolic 3-space. What is the Riemannian metric?

(4) Show that $\mathrm{SL}(2, \mathbb{C})$ acts on H_1 by isometries via $M \cdot A = M\,A\,M^*$. What is the stabilizer of I_2? Recover that $\mathrm{Isom}^+(\mathbb{H}^3) \approx \mathrm{PSL}(2, \mathbb{C})$ and $\mathbb{H}^3 \approx \mathrm{PSL}(2, \mathbb{C})/\mathrm{PSU}(2)$.

Exercise 9. Hyperbolic subspace

Propose a definition of a hyperbolic subspace of a hyperbolic space $X = \mathbb{H}^n$, and describe the hyperbolic subspaces in all the different models of $\mathbb{H}^n$.

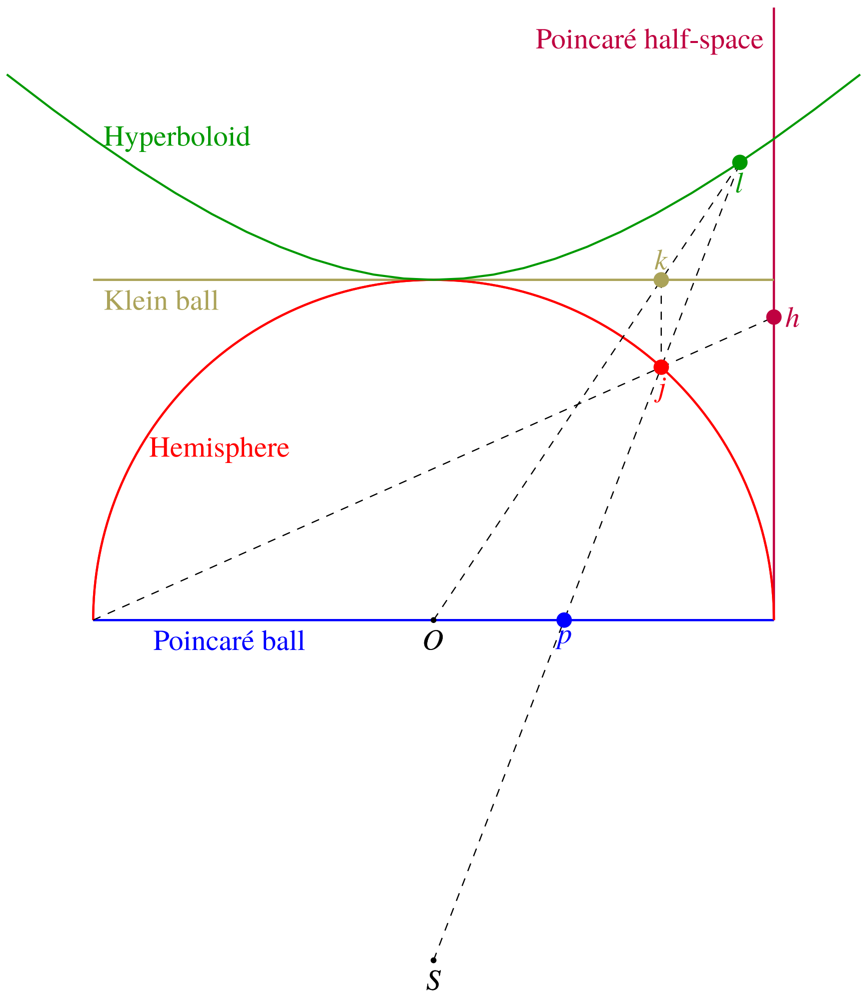

Figure 1: Relation between models of hyperbolic space.

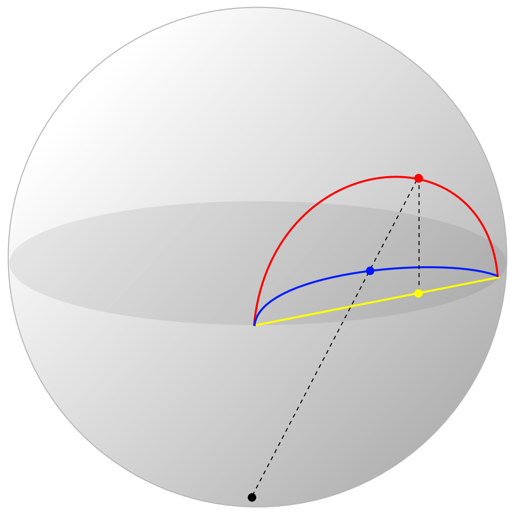

Figure 2: Geodesics in Poincaré ball, Klein ball, and hemisphere models.