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Hyperbolic Geometry

Winter 2019-2020

Exercise Sheet 4 (Chapter 6)

Chapter 6

Exercise 1. Cayley-Klein model of elliptic space

Let (V, b) be a Euclidean vector space. We denote S the unit sphere in V.

(1) Prove Theorem 6.25: *The stereographic projection* $S/\{\pm \mathrm{id}\} \to P(V)$ *is an isometry with respect to the spherical distance on* $S/\{\pm \mathrm{id}\}$ *and the Cayley-Klein metric on* $P(V)$.

(2) Show that the Cayley-Klein metric on $P(V)$ may be written:

$$d([u],[v]) = \arccos\left(\frac{b(u,v)}{\sqrt{b(u,u)b(v,v)}}\right).$$

Exercise 2. Cayley-Klein model of Euclidean space

Let $\mathcal{P} = P(V)$ be a projective space of dimension n and let b be a symmetric bilinear form on V of signature $(1,0)$. Let q denote the associated quadratic form and $Q \subseteq \mathcal{P}$ the associated quadric.

(1) Let b_0 be a Euclidean inner product on $\ker b$. Show that $b_\varepsilon := \varepsilon^2 b_0 + b$ is a Euclidean inner product on V. Write the Cayley-Klein metric d_ε on $P(V)$ associated to b_ε using Exercise 1 (2). Derive the following expression in a suitable affine chart $\mathcal{P} - Q \xrightarrow{\sim} \mathbb{R}^n$:

$$d_\varepsilon(x,y) = \arccos\left(\frac{1+\varepsilon^2\langle x,y\rangle}{\sqrt{(1+\varepsilon^2\|x\|^2)(1+\varepsilon^2\|y\|^2)}}\right).$$

(2) Show that, when $\varepsilon \to 0$, the Cayley-Klein metric d_ε converges to the constant function $d_0 = 0$. Is this expected?

(3) Show that, when $\varepsilon \to 0$, the "blown-up" Cayley-Klein metric $\frac{1}{\varepsilon} d_\varepsilon$ converges to a Euclidean metric on $\mathcal{P} - Q$, which can be identified to b_0. Is this expected?

Exercise 3. Hilbert metric

We have seen that the Cayley-Klein metric d is a distance in $\Omega \subseteq \mathbb{R}^n$ when Ω is the interior of an ellipsoid. Hilbert gave an elegant and elementary proof that applies more generally whenever Ω is a bounded convex open set. Your task is to go and read this proof in [**?**, §5.6], and summarize it in a few lines.

Exercise 4. Beltrami-Klein distance and stereographic projection

(1) Recall the expression of the hyperbolic distance $d_{\mathcal{H}}$ on the hyperboloid $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ and the distance d_{BK} on the Beltrami-Klein disk $B \subseteq \mathbb{R}^n$.

(2) Compute the image of the distance $d_{\mathcal{H}}$ on B under the stereographic projection. Recover that the stereographic projection is an isometry from the hyperboloid to the Beltrami-Klein disk.

Exercise 5. Riemannian metric in the Beltrami-Klein disk

(1) Redo the calculation of the Riemannian metric in the Beltrami-Klein disk (preferably without looking at the lecture notes).

(2) Is the Beltrami-Klein metric conformal to the Euclidean metric in B?

Exercise 6. Distance to origin

Check that the distance from the origin to a point x in the Beltrami-Klein disk $B \subseteq \mathbb{R}^n$ is given by $d(O, x) = \operatorname{artanh}(\|x\|)$, using three different arguments:

(1) Using the expression of the Cayley-Klein metric in terms of cross-ratios.

(2) Using the explicit expression of the distance (see Proposition 6.31).

(3) Using the Riemannian metric.

Exercise 7. Circles in the Beltrami-Klein disk

A *circle* $C(x, R)$ in the 2-dimensional Beltrami-Klein disk (B, d) is the set of points at distance R from x. Show that any circle in the Beltrami-Klein disk is a Euclidean ellipse. Show an analogous result for higher-dimensional Beltrami-Klein disks.

Exercise 8. Geodesics in the Beltrami-Klein disk

Find the expression of any parametrized geodesic in the Beltrami-Klein disk.

Exercise 9. Isometries in the Beltrami-Klein disk

(1) Describe the action of $\mathrm{PO}(1,1)$ on the 1-dimensional Beltrami-Klein disk.

(2) Consider the matrix

$$R(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $R(t) \in \mathrm{SO}(2,1)$ and describe its action on the 2-dimensional Beltrami-Klein disk.

(3) Show that any element of $\mathrm{PSO}(2,1)$ can be written $[L][R]$, for some Lorentz boost L and some $R = R(t)$. (We denote $[M]$ the element of PG associated to $M \in G$.) Recover the fact that $\mathrm{PSO}(2,1)$ is connected.