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Hyperbolic Geometry

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Exercise Sheet 3 (Chapter 5)

Chapter 5

Exercise 1. Projective duality

Let V be a finite-dimensional vector space.

(1) What is projective duality?

(2) If V is equipped with a pseudo-inner product, can you relate taking the orthogonal and taking the annihilator of subspaces of V?

(3) Let P be a projective plane. Prove that any two lines of P intersect at a unique point: first write a direct proof, then propose an alternate proof using projective duality.

Exercise 2. Axioms of projective geometry

Let V be a vector space over a field $\mathbb{K}$. Show that the projective space $\mathbf{P}(V)$ satisfies the axioms of projective geometry (see § 5.1.4).

Exercise 3. First fundamental theorem of projective geometry

Prove Theorem 5.24: *A projective linear map between projective spaces of the same dimension is uniquely determined by the image of a projective frame.*

Exercise 4. Central collineations

Let $\mathcal{P} = \mathbf{P}(V)$ be a projective space of dimension $\geqslant 2$. Given a point O and a projective hyperplane $\mathcal{H}$, a *central collineation* of center O and axis H is a projective transformation $f : \mathcal{P} \to \mathcal{P}$ such that $\mathcal{H}$ is fixed pointwise by f, and any line through O is preserved by f. Let $\hat{f}$ denote the element of $\mathrm{GL}(V)$ associated to f. (Is $\hat{f}$ well-defined?)

(1) Show that f is a central collineation if and only if $\hat{f}$ admits an eigenspace of codimension 1.

(2) A central collineation is called an *elation* if its center belongs to its axis, and a *homology* otherwise. Show that a central collineation f is a homology if and only if $\hat{f}$ is diagonalizable.

(3) Let f be a central collineation of the projective plane with center O. Let l be a line and let $l' = f(l)$. Show that for any point A on l, $A' = f(A)$ is the intersection of the lines l and OA. Comment Figure 1.

(4) In Figure 1, prove that $[A, B, C, D] = [A', B', C', D']$.

(5) (*) Show that every homography is the composition of a finite number of central collineations. *This result is sometimes known as the third fundamental theorem of projective geometry.*

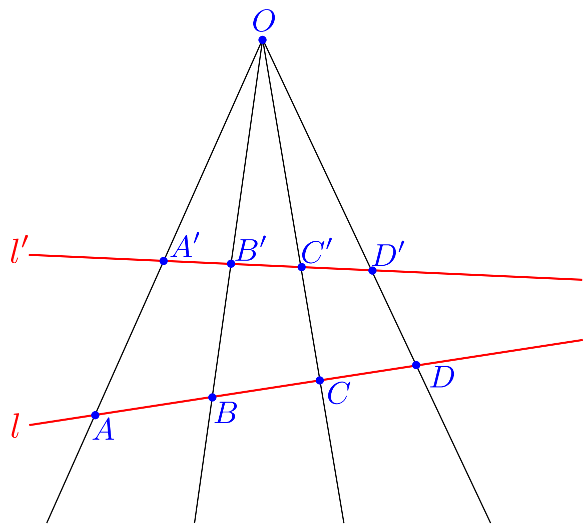

Figure 1: Central collineation in the projective plane.

Exercise 5. Formula for the cross-ratio

Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{K} \cup \{\infty\}$. Check that the map

$$f : z \mapsto \frac{(z - z_2)(z_1 - z_3)}{(z_1 - z)(z_3 - z_2)}$$

is a linear fractional transformation that maps z_1 to ∞, z_2 to 0, and z_3 to 1. Recover the formula for the cross-ratio.

Exercise 6. Cross-ratios and metrology

Consider the photo of Figure 2 (taken from Wikipedia). Denote by A, B, C, D, V the points in the real world, and by A', B', C', D', V' the points in the image. On the photo, one can measure the lengths (in pixels):

$$A'B' = 30\text{px} \quad B'C' = 20\text{px} \quad C'D' = 10\text{px} \quad D'V' = 60\text{px}$$

The goal is to determine the width $w = BC$ (in meters) of the side street.

(1) Justify the equality of cross-ratios $[A, B, C, D] = [A', B', C', D']$. Given the widths of the adjacent shops $AB = 7$m and $CD = 6$m, show that $w = 8$m.

(2) Justify the equality of cross-ratios $[A, B, C, V] = [A', B', C', V']$. Recover the result $w = 8$m using only the width of one adjacent shop $AB = 7$m.

Exercise 7. From a hyperboloid of two sheets to a sphere

Consider the hyperboloid $\mathcal{H}$ of two sheets with equation $x^2 + y^2 - z^2 = -1$ in $\mathbb{R}^3$.

(1) Show that by moving the plane at infinity $\partial_\infty \mathbb{R}^3$, the projective completion of $\hat{\mathcal{H}}$ can be seen as a sphere.

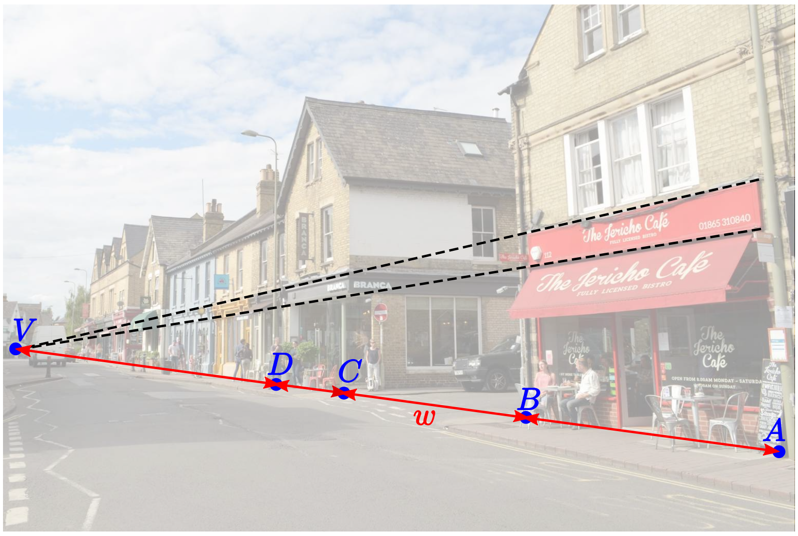

Figure 2: Use of cross-ratios to measure real-world dimensions.

(2) Determine $\partial_\infty \mathcal{H}$ (the intersection of $\hat{\mathcal{H}}$ with the plane at infinity $\partial_\infty \mathbb{R}^3$). Can you describe why $\mathcal{H} \cup \partial_\infty \mathcal{H}$ is a topological sphere?

Exercise 8. Determinant quadric

Let $V = \mathcal{M}_{2\times 2}(\mathbb{K})$ denote the vector space of 2×2 matrices over a field $\mathbb{K}$.

(1) Show that the determinant function $\det \colon V \to \mathbb{K}$ is a quadratic form.

(2) Show that the set of non-invertible matrices defines a nondegenerate quadric in $\mathbf{P}(V)$. Find its normal form when $\mathbb{K} = \mathbb{R}$. *Optional: find an affine chart in which it is a hyperboloid of one sheet, and another where it is a hyperbolic paraboloid.*

(3) Show that $\mathrm{SL}(2, \mathbb{K})$ is an affine quadric in V. What is its projective completion? *Optional: when $\mathbb{K} = \mathbb{R}$, find an affine chart in which it is a hyperboloid of one sheet, and another where it is a hyperbolic paraboloid.*

Exercise 9. Gaussian curvature of quadric surface (*)

Show that the sign of the Gaussian curvature of a surface is a projective invariant. Determine the sign of the Gaussian curvature of the quadric surfaces in normal form.