Brice Loustau
TU Darmstadt

Riemann Surfaces

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Exercises for Chapter 6: Calculus on Riemann surfaces

Exercise 1. Integrable almost complex structures

(1) Let M be a smooth manifold. Show that for any local chart $(x_1, \ldots, x_n)$, we have

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

for all $i, j \in \{1, \ldots, n\}$.

(2) Let M be a complex manifold. Show that any local complex chart $(z_1, \ldots, z_n)$, we have

$$\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right] = 0 \qquad \text{and} \qquad \left[\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}\right] = 0$$

for all $i, j \in \{1, \ldots, n\}$.

(3) Let (M, J) be an almost complex manifold. Recall the definition of $\mathrm{T}^{1,0}M$ and $\mathrm{T}^{0,1}M$.

(4) Let M be a complex manifold. Show that the Lie bracket of vector fields of type $(1, 0)$ (resp. $(0, 1)$) is a vector field of type $(1, 0)$ (resp. $(0, 1)$). One says that $\mathrm{T}^{1,0}M$ and $\mathrm{T}^{0,1}M$ are *integrable*.

(5) Show that the condition that $\mathrm{T}^{1,0}M$ and $\mathrm{T}^{0,1}M$ are integrable is equivalent to the condition that $N(X, Y) = 0$ for any smooth vector fields X and Y on M, where N is the *Nijenhuis tensor* defined by:

$$N(X, Y) = 2\left([JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]\right)$$

Conclude that if (M, J) is a manifold with an integrable almost complex structure, then the Nijenhuis tensor vanishes on M. *The Newlander-Nirenberg theorem states that the converse is also true. It's much harder to prove.*

Exercise 2. Abelian differential criterion

(1) Let X be a Riemann surface and let $f\colon X \to \mathbb{C}$ be a smooth function. Show that f is holomorphic if and only if $\bar{\partial} f = 0$.

(2) Let X be a Riemann surface and let $\alpha \in \mathcal{A}^{1,0}(X, \mathbb{C})$. Show that α is an abelian differential if and only if $\bar{\partial}\alpha = 0$.

(3) Is it true in general, on a complex manifold, that $\alpha \in \mathcal{A}^{p,q}(X, \mathbb{C})$ is holomorphic if and only if $\bar{\partial}\alpha = 0$?

Exercise 3. Functions and abelian differentials on the Riemann sphere

(1) Determine all holomorphic and meromorphic functions on the Riemann sphere.

(2) Determine all abelian differentials on the Riemann sphere.

Exercise 4. Abelian differentials and Dolbeault cohomology on Riemann surfaces

Let X be a Riemann surface. Show that the space of abelian differentials on X identifies to the Dolbeault cohomology space $H^{1,0}(X, \mathbb{C})$.

Exercise 5. Harmonic functions on Riemann surfaces

Let X be a Riemann surface. Recall the definition of being *harmonic* for a smooth function $f : X \to \mathbb{R}$. Show that f is harmonic if and only if f is locally the real part of a holomorphic function.