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Riemann Surfaces

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Exercises for Chapter 4: Surface topology

Exercise 1. Properties of connected sum

Let $\mathcal{S}$ denote the set (?) of topological surfaces up to homeomorphisms. The connected sum # may be seen as an operation on S. What are the properties of this operation?

Exercise 2. Topological invariants

(1) Show that a closed surface and an open surface are not homeomorphic.

(2) Show that an orientable surface and a nonorientable surface are not homeomorphic.

(3) Show that a disk and a punctured disk are not homeomorphic.

(4) A simple closed curve on a surface is called *separating* if removing it disconnects the surface. (*) Show (or admit) that the genus of a closed orientable surface is equal to the maximum number of nonintersecting non-separating simple closed curves. Conclude that closed orientable surfaces of different genera are not homeomorphic.

Exercise 3. Gluing a disk to a Möbius strip

The boundary of the Möbius strip is a topological circle S^1. Therefore one can glue a closed disk to the Möbius strip along their respective boundary. What is the resulting surface?

Exercise 4. Universal cover of the projective plane

Show that the map $x \mapsto -x$ defines an action of the group $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{R}^3$. Show that this action induces a free and wandering group action of $\mathbb{Z}^2$ over the unit sphere S^2. What is the universal cover of the projective plane?

Exercise 5. The Klein bottle

Consider the group Γ of homeomorphisms of $\mathbb{R}^2$ generated by $\tau : (x, y) \mapsto (x + 1, y)$ and $\sigma : (x, y) \mapsto (1 - x, y + 1)$. What is the quotient $K := \mathbb{R}^2/\Gamma$? Try to draw an immersion of K in $\mathbb{R}^3$. How does K fit into the classification of surfaces? *Hint: Show that* $K = \mathbb{R}P^2 \# \mathbb{R}P^2$.

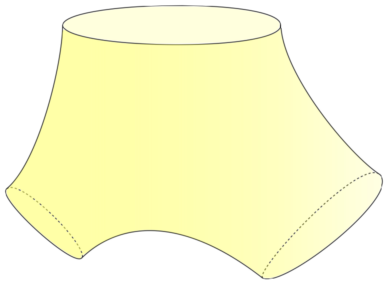

Figure 1: Pair of pants

Exercise 6. Euler characteristic and classification

Is it true that closed surfaces are classified by their Euler characteristic?

Exercise 7. Euler characteristic of S_g

(1) Show (or admit) that for any closed n-manifolds, $\chi(M\#N) = \chi(M)+\chi(N)-\chi(S^n)$. Derive the Euler characteristic of S_g and S_g^{non}.

(2) Consider a topological $4g$-gon P_g, and label its sides a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, $\ldots$, $b_g{}^{-1}$. Glue the sides of P_g according to the labels. What is the resulting surface? *Hint: you may answer the question with an indirect proof: find a triangulation of the surface, compute the Euler characteristic, and use the classification theorem.*.

(3) With hyperbolic geometry, one can show that for any $g \geqslant 2$, there exists a free and wandering discrete group action on the unit disk $\mathbb{D}$, which admits a topological $4g$-gon as fundamental domain, and such that the side identifications under the action of the group are the same as in the previous question. What is the universal cover of S_g?

Exercise 8. Pants decomposition of a closed orientable surface

(1) A *pair of pants* is a surface homeomorphic to the surface drawn in Figure 1. If we consider a pair of pants as an open surface (remove the boundary circles), is it a surface of finite type? How does it fit into the classification of surfaces?

(2) Let S be any closed orientable surface of genus $g \geqslant 2$. Consider a maximal system of non-intersecting simple closed curves on S. Cutting S along these curves yields a finite number of pair of pants, as shown in Figure 2. This is called a *pants decomposition* of S. Without writing a formal proof, guess a formula for the number of curves M and the number of pair of pants N in terms of the genus g.

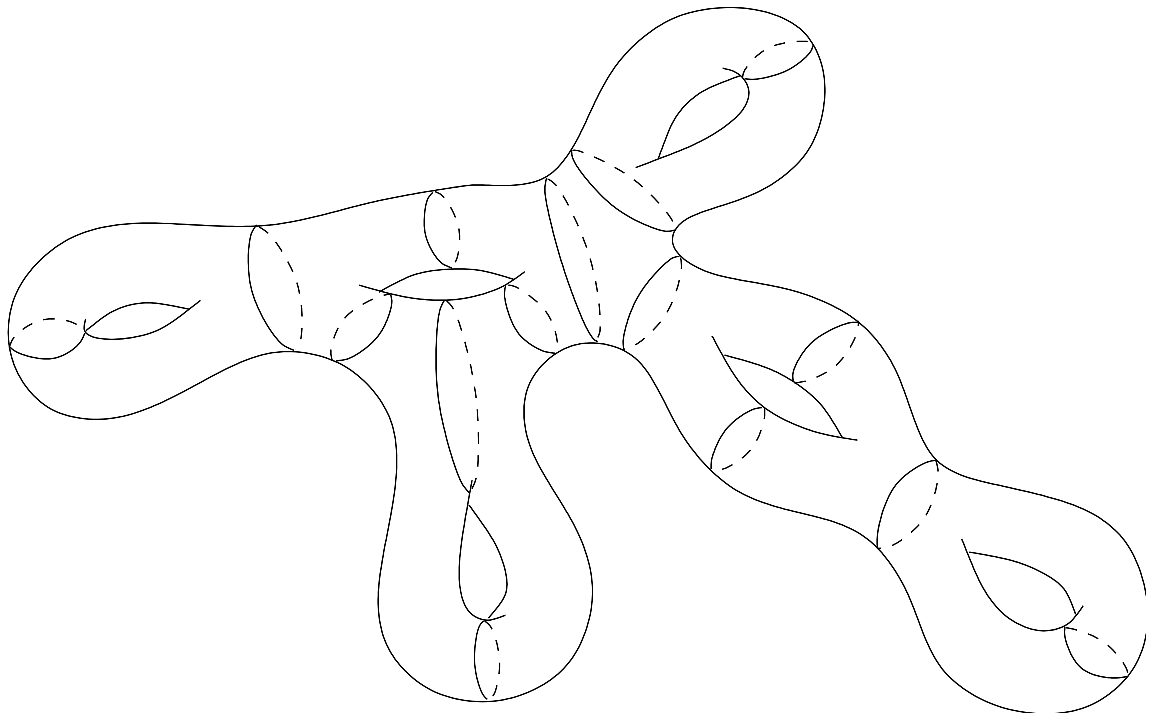

Figure 2: Pants decomposition of a surface