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Riemann Surfaces

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Exercises for Chapter 2: Review of complex analysis

Exercise 1. Complex differentiability ⇔ Real differentiability with $\mathbb{C}$-linear derivative

Identify $\mathbb{R}^2 \approx \mathbb{C}$. Let us denote $J \colon \mathbb{R}^2 \to \mathbb{R}^2$ the map defined by $z \mapsto iz$.

(1) Prove that a linear map $L \colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{C}$-linear if and only if L and J commute.

(2) Write the matrix of J in the standard basis of $\mathbb{R}^2$. Characterize the matrix of a linear map $L \colon \mathbb{R}^2 \to \mathbb{R}^2$ that is $\mathbb{C}$-linear.

(3) Let $f \colon U \subset \mathbb{C} \to \mathbb{C}$ a function that is real-differentiable at $z_0 \in \mathbb{C}$. Show that $\mathrm{d}_{z_0} f$ is $\mathbb{C}$-linear if and only if the partial derivatives of f at z_0 satisfy the Cauchy-Riemann equations.

(4) Let $a \in \mathbb{C}$, write the matrix of the linear map $M_a \colon z \mapsto az$. Show that a linear map $L \colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{C}$-linear if and only if there exists $a \in \mathbb{C}$ such that $L = M_a$.

(5) Let $f \colon U \subset \mathbb{C} \to \mathbb{C}$ a function that is real-differentiable at $z_0 \in \mathbb{C}$. Show that f is complex-differentiable at z_0 if and only if $\mathrm{d}_{z_0} f$ is $\mathbb{C}$-linear. Moreover, $\mathrm{d}_{z_0} f = M_{f'(z_0)}$.

Exercise 2. Holomorphic ⇔ Conformal

(1) Let $L \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Show that the following are equivalent:
  - (i) L preserves oriented angles between vectors. *Start by rephrasing this condition more precisely.*
  - (ii) L is a similitude. *Remind yourself what a similitude is.*
  - (iii) L is $\mathbb{C}$-linear.

(2) Let $f \colon U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is an open set. Show that the following are equivalent:
  - (i) f preserves oriented angles between curves. *Start by rephrasing this condition more precisely. By definition, this condition says that f is conformal.*
  - (ii) $\mathrm{d}f$ preserves angles between oriented vectors.
  - (iii) f is holomorphic.

Exercise 3. $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ operators

Let $f \colon U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is an open set. Identify $\mathbb{C} \approx \mathbb{R}^2$.

(1) Do you remember the expression of the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$? If not, try to recover it, knowing that the identity $\mathrm{d}f = \frac{\partial f}{\partial z}\,\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z}$ must be true for any real-differentiable function f, in particular $f(z) = z$ and $f(z) = \bar{z}$.

(2) Prove that $\mathrm{d}f = \frac{\partial f}{\partial z}\,\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z}$ for any differentiable function f.

(3) Prove that the Cauchy-Riemann equations for f are equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$. Conclude that f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.

(4) Show that the following are equivalent:
  (i) $z \mapsto f(\bar{z})$ is holomorphic.
  (ii) $z \mapsto \overline{f(z)}$ is holomorphic
  (iii) f is *anticonformal*: f reverses oriented angles between curves.
  (iv) $\frac{\partial f}{\partial z} = 0$.
  NB: A function that satisfies these conditions is called antiholomorphic

(5) Example: let $f(z) = z^3 + \bar{z}^7 + 3z^2\bar{z}^2$. Is f holomorphic? Is f antiholomorphic?

Exercise 4. Holomorphy and harmonicity

Let $f\colon U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is an open set. Recall that the Laplacian Δ is the operator defined by $\Delta f = \operatorname{tr}(\operatorname{Hess} f)$, where the trace is taken in any orthonormal basis. This definition works for real- or complex-valued functions. The function f is called *harmonic* if $\Delta f = 0$.

(1) Show that
$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial \bar{z} \partial z} = \frac{1}{4}\Delta$$

(2) Show that if f is holomorphic or antiholomorphic, then f is harmonic.

(3) Is the converse true? *Hint: if f_1 and f_2 are real-valued harmonic functions, then $f = f_1 + i f_2$ is still harmonic.*

(4) Show that if $f_1 : U \to \mathbb{R}$ is harmonic, then f_1 is locally the real part of a holomorphic function.

(5) Example: let $f_1(x, y) = 2xy$. Find a holomorphic function with real part f_1.

Exercise 5. Logarithm and nth roots of a holomorphic function

Let $f\colon U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is a connected open set. One calls *logarithm of f* any holomorphic function $g\colon U \to \mathbb{C}$ such that $\exp \circ g = f$.

(1) Show that g is a logarithm of f if and only if $g' = \frac{f'}{f}$ and $g(0) = \exp(f(0))$.

(2) Show that if U is simply connected, then f admits a logarithm if and only if f does not vanish in U. Show that any two logarithms differ by an integer multiple of $2i\pi$.

(3) Let $f: U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is a simply connected open set. Assume f does not vanish in U. How would you define a *square root of* f? How many square roots of f are there? Same question of nth roots of f.

Exercise 6. Local structure of a holomorphic function near a zero

Let $f: U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is an open set. A point $z_0 \in U$ is called a *zero of order n of f*, where $n \in \mathbb{N}^*$, if $f^{(k)}(z_0) = 0$ for all $k \in \{1, \ldots, n-1\}$ and $f^{(n)}(z_0) \neq 0$.

(1) Characterize the fact that z_0 is a zero of order n of f in terms of the coefficients of the power series representing f at z_0.

(2) Show that z_0 is a zero of order n of f if and only if there exists a holomorphic function $g: U \to \mathbb{C}$ such that $f(z) = (z - z_0)^n g(z)$ and $g(z_0) \neq 0$.

(3) Show that z_0 is a zero of order n of f if and only if there exists a neighborhood $V \subseteq U$ of z_0 and a holomorphic function $h: V \to \mathbb{C}$ with a simple zero (zero of order 1) at z_0 such that $f = h^n$ in V. *Hint: show that g admits a nth root in a small disk centered at z_0.*

(4) Show that if z_0 is a zero of order n of f, then there exists a neighborhood V of z_0 in U and a neighborhood W of $f(z_0)$ in $\mathbb{C}$ such that every element of $W - \{f(z_0)\}$ has exactly n preimages in $V - \{z_0\}$.

Exercise 7. An application of Liouville's theorem

What can you say about two entire functions f and g such that $|f| < |g|$ on $\mathbb{C}$? Show that the result remains true if $|f| \leqslant |g|$.

Exercise 8. An extension of Liouville's theorem

Let f be an entire function. Show that the image of f is either a point or it is dense in $\mathbb{C}$. *Hint: by contradiction, assume f misses a disk $D(z_0, r)$. Post-compose f with an appropriate function so that the resulting map misses a "disk centered at ∞".*

Exercise 9. Application of a theorem

What can you say about a real-valued holomorphic function? What about a holomorphic function with constant modulus? *Hint: the answer is an immediate consequence of one of the essential theorems from the lectures.*

Exercise 10. Singularities of holomorphic functions: examples

Classify the singularities of the following functions. Give the order of the poles.

(1) $z \mapsto \dfrac{z^4}{(z^4+16)^2}$

(2) $z \mapsto \dfrac{1-\cos z}{\sin z}$

(3) $z \mapsto \dfrac{z}{e^z - z + 1}$

(4) $z \mapsto \dfrac{z^2-\pi^2}{\sin z}$

(5) $z \mapsto \dfrac{1}{e^z-1} - \dfrac{1}{z-2\pi i}$

(6) $z \mapsto \dfrac{1}{\cos(1/z)}$

Exercise 11. The ring $\mathcal{H}(U)$ and the field $\mathcal{M}(U)$

Let $U \subseteq \mathbb{C}$ be a connected open set. We denote $\mathcal{H}(U)$ the set of holomorphic functions on U and $\mathcal{M}(U)$ the set of meromorphic functions on U. Show that $\mathcal{H}(U)$ is an integral domain and that $\mathcal{M}(U)$ is a field isomorphic to the fraction field of $\mathcal{H}(U)$.

Exercise 12. Automorphisms of $\mathbb{C}$

Determine $\mathrm{Aut}(\mathbb{C})$.

Exercise 13. Automorphisms of $\mathbb{D}$ and $\mathbb{H}$ (*)

For $a \in \mathbb{D}$, let us denote $\varphi_a : \mathbb{D} \to \mathbb{C}$ the map defined by $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$.

(1) Show that φ_a is well-defined and that it is an automorphism of $\mathbb{D}$. Find its inverse.

(2) Show that any automorphism f of $\mathbb{D}$ is of the form $u\varphi_a$, where u is a unit complex number and $a \in \mathbb{D}$. *Hint: consider $\varphi_{f(a)} \circ f$ and use the Schwarz lemma.*

(3) Define $\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : \ (a,b) \in \mathbb{C}^2, \ |a|^2 - |b|^2 = 1 \right\}$ and $\mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\pm I_2$. Show that $\mathrm{Aut}(\mathbb{D}) \approx \mathrm{PSU}(1,1)$.

(4) Let $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half-plane. Find a Riemann mapping $\mathbb{H} \to \mathbb{D}$. *Hint: try the Cayley map $z \mapsto \frac{z-i}{z+i}$.* Show that $\mathrm{Aut}(\mathbb{H}) \approx \mathrm{PSL}(2,\mathbb{R})$.

Exercise 14. Automorphisms of a punctured open set (*)

(1) Let U be a connected open set, let $z_0 \in U$, and let $f\colon U - \{z_0\} \to \mathbb{C}$ be a holomorphic function. Show that if f is injective, then z_0 is a removable singularity or a pole. Show that if it is a removable singularity, the holomorphic extension of f is still injective. Show that if it is a pole, it is a simple pole (pole of order 1).

(2) Let $f\colon \mathbb{C}^* \to \mathbb{C}^*$ be injective. Show that $f(z) = az$ or $f(z) = \frac{a}{z}$, with $a \in \mathbb{C}^*$. What is the automorphism group $\mathrm{Aut}(\mathbb{C}^*)$?

(3) Let U be a bounded connected open set. Let $z_0 \in U$, denote $U^* = U \setminus \{z_0\}$. Assume that U^* has no other punctures: for every $a \in \delta U$, $U \cup \{a\}$ is not open. Show that every automorphism of U^* coincides with an automorphism of U that fixes z_0.

(4) Describe $\mathrm{Aut}(\mathbb{D}^*)$. (We denote $\mathbb{D}^* := \mathbb{D} - \{0\}$).