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General Relativity

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Exercise Sheet 4

Exercise 1. Full Einstein-Hilbert functional

Let M be a compact orientable smooth manifold.

(1) Recall that in the vacuum and with no cosmological constant, the Einstein-Hilbert functional is:

$$\mathcal{S}(g) = \frac{1}{2\kappa}\int_M S_g\, v_g$$

where g is any semi-Riemannian metric and we denote by S_g and v_g the scalar curvature and volume form of g, and $\kappa = \frac{8\pi G}{c^4} = 8\pi$ is a constant. Recall why the Euler-Lagrange equation for this action is $\mathrm{Ric} - \frac{1}{2}Sg = 0$.

(2) In the presence of a cosmological constant $\Lambda \in \mathbb{R}$, the Einstein-Hilbert functional is

$$\mathcal{S}(g) = \frac{1}{2\kappa}\int_M (S_g - 2\Lambda)\, v_g \ .$$

Show that the Euler-Lagrange equation is $\mathrm{Ric} - \frac{1}{2}Sg + \Lambda g = 0$.

(3) (*) The presence of matter is encoded by a smooth real-valued function $\mathcal{L}_{\mathrm{M}}(g,x)$ which depends on g and $x \in M$ (the *Einstein-Hilbert Lagrangian*). In this case the E-H functional is

$$\mathcal{S}(g) = \int_M \left[\frac{1}{2\kappa}(S_g - 2\Lambda) + \mathcal{L}_{\mathrm{M}}\right] v_g \ . \tag{1}$$

Show that the Euler-Lagrange equation is

$$\mathrm{Ric} - \frac{1}{2}Sg + \Lambda g = \kappa T$$

where $T := -2\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} + \mathcal{L}_{\mathrm{M}} g_{\mu\nu}$ is the *stress-energy tensor*. (Equation (1) is called *Einstein's field equations*.) How to interpret a solution of this equation?

Exercise 2. Second variation of the Einstein-Hilbert functional

Let g be a critical point of the Einstein-Hilbert functional $\mathcal{S}(g) = \int_M S_g\, v_g$, i.e. a Ricci-flat metric. The second variation of the E-H functional in the direction of a symmetric covariant 2-tensor h is $\mathcal{S}''_g(h) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0}\mathcal{S}(g+th)$.

(1) Show that $\mathcal{S}$ has no strict local extrema by consider constant scaling of the metric . For this reason, we now only consider variations which preserve the total volume.

(2) Show that $\mathrm{Diff}(M)$ naturally acts on metrics by pullback and that $\mathcal{S}$ is constant in restriction to any orbit. Conclude that $\mathcal{S}''_g(h) = 0$ if h is tangent to the $\mathrm{Diff}(M)$-orbit through g. Describe such tensors h.

(3) Consider the conformal class of g, i.e. the space of metrics of the form fg for some smooth function $f\colon M \to (0,+\infty)$. Show that tangent deformations to this space are of the form $h = fg$ for some smooth function $f\colon M \to \mathbb{R}$, and the infintesimal variation preserves volume if and only if $\int_M f\, v_g = 0$. (*) For such h, show that the second variation is $\mathcal{S}''_g(h) = -\frac{(n-1)(n-2)}{2}\int_M f\Delta f\, v_g$. Conclude that g is a strict local minimum of $\mathcal{S}$ in restriction to such conformal deformations.

Exercise 3. Hodge star, codifferential, divergence, and Hodge Laplacian

Let (M,g) be a compact oriented semi-Riemannian manifold of dimension n.

(1) Consider a fixed tangent space $V = \mathrm{T}_x M$ with inner product $g_x = \langle \cdot, \cdot \rangle$. Show that there is a natural inner product in $\Lambda^k V^*$. (First define an inner product in V^*, then in the space of k-multilinear maps.) We define an inner product on $\Omega^k(M,E)$ by $\langle \alpha, \beta \rangle_{\mathrm{L}^2} := \int_M \langle \alpha, \beta \rangle \, v_g$.

(2) The *Hodge star* is the operation $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ characterized by $\alpha \wedge *\beta = \langle \alpha, \beta \rangle v_g$.
  (i) Show that $*$ is well-defined and express it using an orthonormal frame of V.
  (ii) Show that $*1 = v_g$.
  (iii) Show that the Hodge star is a linear isometry: $\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle$.
  (iv) Show that the Hodge star is an involution up to sign: $**\alpha = (-1)^{k(n-k)+\mathrm{index}(g)} \alpha$.

(3) Define the *codifferential* $\mathrm{d}^* := (-1)^{n(k-1)+1+\mathrm{index}(g)} * \mathrm{d} *$.
  (i) Show that d^* is a linear map $: \Omega^k(M,\mathbb{R}) \to \Omega^{k-1}(M,\mathbb{R})$ for any $k \in \{0,\dots,n\}$.
  (ii) Check that $\mathrm{d}^* \circ \mathrm{d}^* = 0$.
  (iii) Show that d^* is the formal adjoint of the differential d: $\langle \mathrm{d}\alpha, \beta \rangle_{\mathrm{L}^2} = \langle \alpha, \mathrm{d}^*\beta \rangle_{\mathrm{L}^2}$.

(4) The *divergence* of a vector field X is the function $\operatorname{div} X$ defined by: $\mathrm{d}(i_X v_g) = (\operatorname{div} X) v_g$ where $i_X v_g \in \Omega^{n-1}(M,\mathbb{R})$ is the contraction of X against v_g.
  (i) Show that $\operatorname{div} X = -\mathrm{d}^* X$.
  (ii) Prove the divergence theorem: $\int_M (\operatorname{div} X) \, v_g = 0$.

(5) The *Hodge Laplacian* is the operator $\Delta := \mathrm{d}^* \mathrm{d} + \mathrm{d}\,\mathrm{d}^*$.
  (i) Show that Δ is an endomorphism of $\Omega^k(M,\mathbb{R})$ for any $k \in \{0,\dots,n\}$.
  (ii) Show that on $\Omega^0(M,\mathbb{R})$, the Hodge Laplacian is equal to minus the Laplace-Beltrami operator defined by $\Delta f = \operatorname{div}(\operatorname{grad} f)$.
  (iii) Show that if g is Riemannian, the Hodge Laplacian is a nonnegative operator in the sense that $\langle \Delta\alpha, \alpha \rangle_{\mathrm{L}^2} \geqslant 0$ and that show that $\langle \Delta\alpha, \alpha \rangle_{\mathrm{L}^2} = 0$ if and only if $\Delta\alpha = 0$. Show that α is harmonic ($\Delta\alpha = 0$) iff α is closed and co-closed ($\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$).

Exercise 4. Killing fields

Let (M,g) be a compact semi-Riemannian manifold. A smooth vector field X is called a *Killing fied* if $\mathcal{L}_X g = 0$, where $\mathcal{L}$ denotes the Lie derivative.

(1) Recall the definition(s) of the Lie derivative.

(2) Show that X is a Killing field if and only if the flow of X preserves g: the diffeomorphism φ_t^X is an isometry for all t. Why did we assume M is compact?

(3) Show that X is a Killing field if and only if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all vector fields Y and Z, where ∇ is the Levi-Civita connection of g.

(4) Let M be a Minkowski spacetime and let $\xi = (t,x,y,z)$ be an inertial coordinate system. (What's that again?) Show that $e_1 = \frac{\partial}{\partial t}$ is a Killing field.

(5) Show that any parallel vector field is a Killing field.

(6) (*) Conversely, let X be a Killing field. Assume (M,g) has nonpositive Ricci curvature. Derive from Bochner's formula $-\frac{1}{2}\Delta\|X\|^2 = -\operatorname{Ric}(X,X) + \|\nabla X\|^2$ that X is parallel. Show that if (M,g) has negative Ricci curvature then it admits no Killing fields other than 0.

(7) (*) Show that if X is a Killing field and α is a harmonic form then $\mathcal{L}_X \alpha = 0$.