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General Relativity

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Exercise Sheet 3 Exercise 6: Solution

Exercise 6. Minimal hypersurfaces

(1) By definition of the pullback metric $g_t = (f_t)^* g$, for all $p \in M$ and for all $X \in \mathrm{T}_p M$:

$$g_t(X, X) = \langle \mathrm{d}f_t(X), \mathrm{d}f_t(X) \rangle \ .$$

Therefore we find

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g_t(X, X) = 2 \left\langle \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{d}f_t(X), \mathrm{d}f(X) \right\rangle \tag{1}$$

I claim that for all $X \in \mathrm{T}_p M$, $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{d}f_t(X) = \nabla_{\mathrm{d}f(X)} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f_t(p)$. Why is that true? It's essentially saying that the t-derivative and the X-derivative of f_t commute: $\nabla_X \nabla_{\partial_t} = \nabla_{\partial_t} \nabla_X$. I'll let you figure out a proper justification.
In the present case, we have $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f_t(p) = \dot{r}(p) \vec{N}(p)$, so the previous observation yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{d}f_t(X) &= \nabla_{\mathrm{d}f(X)} \left(\dot{r}(p) \vec{N}(p) \right) \\ &= (\mathrm{d}\dot{r}(X)) \vec{N}(p) + \dot{r}(p) \nabla_{\mathrm{d}f(X)} \vec{N}(p) \ . \end{aligned}$$

We thus derive from (1):

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g_t(X, X) = 2 \left\langle (\mathrm{d}\dot{r}(X)) \vec{N}(p), \mathrm{d}f(X) \right\rangle + 2 \left\langle \dot{r}(p) \nabla_{\mathrm{d}f(X)} \vec{N}(p), \mathrm{d}f_t(X) \right\rangle \ . \tag{2}$$

Note that $\mathrm{d}f_t(X)$ is tangent to the surface and $\vec{N}$ is normal to it, so the first term of (2) is zero. It remains:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g_t(X, X) = 2\dot{r}(p) \left\langle \nabla_{\mathrm{d}f(X)} \vec{N}(p), \mathrm{d}f(X) \right\rangle$$

in other words $\dot{g}(X, X) = -2\dot{r} b(X, X)$, which is what we wanted. (Of course, one immediately gets $\dot{g}(X, Y) = -2\dot{r} b(X, Y)$ by polarization.)

(2) This is an immediate consequence of the previous question and the following general fact, which you saw in the lecture:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{vol}(g_t) = \frac{1}{2} \mathrm{tr}_{g_0}(\dot{g}) \, \mathrm{vol}(g_0)$$

(3) By the previous question,

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{A}(f_t) = \int_S -\dot{r}H \,\mathrm{vol}(g_0) \ .$$

Clearly, this is zero if $H = 0$. Conversely, imagine that $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{A}(f_t) = 0$ for any choice of $r(t,p)$ (with always $r(0,p) = 0$). In particular, one can choose a function r such that $-\dot{r} = H$: for example, take $r(t,x) = -H(x)t$. Then item (3) reads $0 = \int_S H^2 \,\mathrm{vol}(g_0)$, which clearly implies $H = 0$.

(4) Clearly, if $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{A}(f_t) = 0$ for any variation, then this is true in particular for normal variations, so $H = 0$ by the previous question. Let us now prove the converse.
Call V the vector field tangent to the variation:

$$V(p) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f_t(p) \ .$$

A useful preliminary remark is that $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{A}(f_t)$ only depends on V, not on the actual variation (f_t).
In any case, the same argument as before shows that $\dot{g}(X,X) = 2\langle \nabla_X V, \mathrm{d}f(X)\rangle$, and that $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathrm{vol}(g_t) = 0$ if V is normal (under the assumption that $H = 0$).
Considering the decomposition of V into tangential and normal components to $f : S \longrightarrow M$, in order to conclude it is enough to show that $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{A}(f_t) = 0$ for tangential vector fields V. If V is tangent to S, since f is an immersion, there exists a unique vector field W on S such that $V = \mathrm{d}f(W)$. However in this case there exists a one-parameter family (φ_t) of diffeomorphisms of S such that $W = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\varphi_t$ (for instance φ_t can be the flow of W). One can then take $f_t = f \circ \varphi_t$ for the variation such that $V = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f_t$. But then it is easy to argue that $\mathcal{A}(f_t) = \mathcal{A}(f)$ (by change of variables) is constant, therefore $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{A}(f_t) = 0$.