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General Relativity

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Exercise Sheet 3

Exercise 1. Second Bianchi identity

The goal of this exercise is to prove the second Bianchi identity:

$$(\nabla_U R)(X,Y,V,W) + (\nabla_V R)(X,Y,W,U) + (\nabla_W R)(X,Y,U,V) = 0\ .$$

(1) Show that the second Bianchi identity can equivalently be written

$$(\nabla_U R)(V,W,X,Y) + (\nabla_V R)(W,U,X,Y) + (\nabla_W R)(U,V,X,Y) = 0\ . \tag{1}$$

(2) Our proof strategy is to show that (1) holds at any fixed point $p \in M$. Argue carefully that we can assume that the vector fields X, Y, U, V, W are all elements of a local orthonormal frame $(E_i)_{1\leqslant i\leqslant m}$ and that their covariant derivatives all vanish at p: $\nabla_X Y = 0$, etc.

NB: This is a recurring trick in Riemannian geometry, also used in the course notes.

(3) Under the previous assumptions, show that the first term of (1) can be written:

$$(\nabla_U R)(V,W,X,Y) = \langle \nabla_U \nabla_V \nabla_W X - \nabla_U \nabla_W \nabla_V X\ ,\ Y\rangle\ .$$

This is obviously zero, since $\nabla_W X = 0$ and $\nabla_V X = 0$. We can treat the other terms of (1) similarly, so we are done. Right? *No!*

(4) Show that the left-hand side of (1) can be rewritten like below and conclude.

$$\langle R(U,V)(\nabla_W X) + R(V,W)(\nabla_U X) + R(W,U)(\nabla_V X)\ ,\ Y\rangle$$

Exercise 2. Einstein equations in the vacuum with cosmological constant

Let $\Lambda \in \mathbb{R}$ be a constant. Show that a semi-Riemannian manifold (M,g) of dimension $m \neq 2$ satisfies the Einstein equation $\mathrm{Ric}(g) - \frac{1}{2}Sg + \Lambda g = 0$ if and only if $\mathrm{Ric}(g) = \lambda g$ where $\lambda = \frac{2\Lambda}{m-2}$ is a constant. *NB: This is called an Einstein manifold (see Exercise 4).* What about the case $\dim M = 2$?

Exercise 3. Schur's lemma

Let (M,g) be a connected semi-Riemannian manifold of dimension $\neq 2$ such that $\mathrm{Ric}(g) = \lambda g$ for some differentiable function $\lambda\colon M \to \mathbb{R}$. Show that λ is constant (in other words (M,g) is Einstein: see Exercise 4). *Hint: take the divergence of both sides of* $\mathrm{Ric}(g) = \lambda g$ *to show that* $\frac{1}{2}\,\mathrm{d}S = \mathrm{d}\lambda$.

Exercise 4. Constant sectional curvature vs Einstein manifold

A semi-Riemannian manifold (M,g) is called an *Einstein manifold* if it has constant Ricci curvature in the sense that there exists a constant $\lambda \in \mathbb{R}$ such that $\mathrm{Ric}(g) = \lambda g$.

(1) Show that any semi-Riemannian manifold (M,g) with constant sectional curvature is Einstein.

(2) Show that the converse of (1) is true when $\dim M = 2$.

(3) (*) Show that the converse of (1) is true when $\dim M = 3$. *Do Exercise 5!*

(4) (*) Is the converse of (1) always true? Consider $\mathbb{C}P^n$ with the Fubini-Study metric.

Exercise 5. Einstein manifolds of dimension 3.

Let (M, g) be a semi-Riemannian manifold of dimension 3. If (M, g) has constant sectional curvature, then (M, g) is Einstein: see Exercise 4. The goal of this exercise is to show the converse.

(1) Let (e_1, e_2, e_3) be an orthonormal basis of $\mathrm{T}_p M$ for some $p \in M$. Show that:

$$K(e_1, e_2) + K(e_1, e_3) = \mathrm{Ric}(e_1, e_1)$$
$$K(e_1, e_2) + K(e_2, e_3) = \mathrm{Ric}(e_2, e_2)$$
$$K(e_1, e_3) + K(e_2, e_3) = \mathrm{Ric}(e_3, e_3)$$

(2) Derive from the previous question that the sectional curvature of any 2-plane through p can be computed in terms of the Ricci curvature. Conclude.

Exercise 6. Minimal hypersurfaces

Let S be a compact oriented manifold of dimension n, let (M, g) be a semi-Riemannian manifold of dimension $n + 1$, and let $f : S \hookrightarrow M$ be a smooth embedding (or immersion).

(1) Let $(f_t)_{t \in I} : S \to M$ be a one-parameter family of maps with $f_0 = f$. Denote $g_t := (f_t)^* g$ (*first fundamental form* of f_t). Let us assume that (f_t) is a *normal variation* of f:

$$f_t(x) = \exp_{f(x)} \left(r_t(x) \vec{N}(x) \right)$$

where $\vec{N}$ is the unit normal vector field to $f : S \hookrightarrow M$[1] and r is some smooth function. Show:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g_t = -2\dot{r} b$$

where $\dot{r}(x) := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} r_t(x)$ and b is the *second fundamental form* of f, defined by:

$$b(X, Y) = -\langle \nabla_{\mathrm{d}f(X)} \vec{N}, \mathrm{d}f(Y) \rangle$$

where ∇ is the L-C connection of (M, g). *Feel free to take* $(M, g) = \mathbb{R}^{n+1}$ *for this question.*

(2) Derive from the previous question that for any normal variation f_t as above, denoting $\mathrm{vol}(g_t)$ the volume form of g_t:

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathrm{vol}(g_t) = -\dot{r} H(f) \, \mathrm{vol}(g_0)$$

where $H := \mathrm{tr}_{g_0} b$ is called the *mean curvature* of the embedding $f : S \to M$.

(3) Define the *area functional* $\mathcal{A}$ by

$$\mathcal{A}(f) = \int_S \mathrm{vol}(f^* g)$$

for any smooth immersion $f : S \to M$. Show that f has vanishing mean curvature ($H = 0$) if and only if $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathcal{A}(f_t) = 0$ for any normal variation (f_t) of f.

(4) (*) Show that f has vanishing mean curvature iff $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathcal{A}(f_t) = 0$ for any variation (f_t).
We proved that $H = 0$ *if and only if* f *is a critical point of the area functional. Such an immersion is called a minimal hypersurface.*

[1] The fact that both S and M are orientable implies that $S \hookrightarrow M$ is two-sided. A choice of unit normal vector field is then determined by a choice of orientation of both M and N.