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General Relativity

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Exercise Sheet 2

Exercise 1. Comparing clocks

In mathematical terms, explain why the clock of a freely falling observer runs faster than the clock of any other observer. How does this relate to the twin paradox?

Exercise 2. Newtonian coordinate systems

Let M be a Minkowski spacetime. Recall that a *Lorentz* or *inertial* coordinate system is a (local) coordinate system $\xi = (t, x, y, z)\colon M \to \mathbb{R}^4_1$ such that ξ is an isometry.

Given a freely falling observer o with normalized proper time τ, a *Newtonian coordinate system* is an inertial coordinate system $\xi = (t, x, y, z)$ such that $\xi(o(\tau)) = (\tau, 0, 0, 0)$.

(1) Given a freely falling observer o, how unique is a Newtonian coordinate system?

(2) What about existence?

(3) Does it matter that the observer is freely falling? Does it matter that M is a Minkowski spacetime and not any Lorentzian manifold?

Exercise 3. Failure of simultaneity

Let o be a freely falling observer in a Minkowski spacetime M. Two events p and q are considered *simultaneous* from the point of view of o if they have the same t-coordinate in any/some Newtonian coordinate system relative to o.

(1) Show that two events p and q are simultaneous from the point of view of o if and only if $\vec{pq}$ is orthogonal to o (i.e. p and q are in the same restspace relative to o).

(2) Show that the notion of simultaneity is not independent of the observer. In fact, give a necessary and sufficient condition for two observers to agree on simultaneity.

Exercise 4. Length contraction

Consider an unaccelerated train travelling at a fraction v of the speed of light relative to some observer o. If d is the length of the train from the point of view of a sitting passenger, what is the length of the spaceship from the point of view of o? Answer the questions below.

(1) Let o_1 and o_2 be two freely falling particles in a Minkowski spacetime M, representing the endpoints of the train. Show that o_1 and o_2 are parallel in M if and only if they are parallel in the restspace of some/any freely falling observer.

(2) From now on we assume that o_1 and o_2 are parallel. Assume that o sits on the train. Show that the distance L between o_1 and o_2 in the restspace of o is constant and equal to their distance in M.

(3) Now let o be any freely falling observer. Assume that o, o_1, and o_2 are coplanar in M. What does this assumption mean physically? From now on we thus assume that $\dim M = 2$.

(4) Let θ denote the hyperbolic angle between o and o_1. Draw o, o_1, and o_2 in an inertial coordinate system relative to o. Show that the distance L_o between o_1 and o_2 is constant, and by doing hyperbolic trigonometry in the right triangle, verifies $L = L_o \cosh\theta$. Conclude that $L_o = L\sqrt{1-v^2}$ where v is the relative speed of the train with respect to o.

Exercise 5. Velocity-addition formula for collinear motions

(1) (a) For $\theta \in \mathbb{R}$, consider the hyperbolic rotation matrix

$$R_\theta = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Show that R_θ is an isometry of Minkowski space $\mathbb{R}^4_1$, in fact show that it is a Lorentz boost: cf Ex. 2 in Exercise Sheet #1. What is $R_{\theta_2} \circ R_{\theta_1}$?

(b) Let V be a normalized timelike vector in Minkowski space $\mathbb{R}^4_1$. Assume V is contained in the tx-plane. Call θ the hyperbolic angle $\theta = \angle(e_0, V)$. Show that R_θ is the only orthochronous isometry $\mathbb{R}^4_1$ fixing the yz-plane such that $V = R_\theta(e_0)$.

(2) Let M be a Minkowski spacetime. Consider three observers o, o_1, and o_2, all freely falling, through some event $p \in M$.

(a) From now on we assume that o, o_1, and o_2 are coplanar. Show that this amounts to saying that seen from o, the Newtonian motions of o_1 and o_2 are collinear.

(b) Explain why, by choosing an appropriate inertial coordinate system, one can assume that $M = \mathbb{R}^4_1$, $p = 0$, $o(\tau) = (\tau, 0, 0, 0)$, and o_1 and o_2 are contained in the tx-plane.

(c) Call V, V_1, and V_2 the initial tangent vectors to o, o_1, and o_2 respectively. Denote the following oriented hyperbolic angles: $\theta_1 = \angle(V, V_1)$, $\theta_2 = \angle(V_1, V_2)$, and $\theta = \angle(V, V_2)$. Show that $R_{\theta_1}(o) = o_1$, $R_{\theta_2}(o_1) = o_2$, and $R_\theta(o) = o_2$. Show that $\theta = \theta_1 + \theta_2$.

(d) Call v_1 (resp. v_2) the signed relative speed of o_1 (resp. o_2) from the point of view of o (resp. o_1). Show that the signed relative speed of o_2 from the point of view of o is:

$$v = \frac{v_1 + v_2}{1 + v_1 v_2} =: v_1 \oplus v_2 .$$

(e) Is $\oplus$ a commutative operation on $\mathbb{R}$? Is it associative? Is it $\mathbb{R}$-linear?

Exercise 6. An accelerated observer

Let $M = \mathbb{R}^2_1$ and let $o(\tau) = (\tau, 0)$ be a freely falling observer. Consider the curve $\alpha(\tau) = (\sinh \tau, \cosh \tau)$.

(1) Show that α is a material particle. Is it freely falling? Show that the norm of its acceleration is constant.

(2) Show that a light beam emitted from α towards o reaches o in finite proper time, but never reaches back α after it is reflected. Hence *o never appears on α's radar*.

(3) Which events can be picked up on α's radar?

(4) Consider a photon being emitted by α at $\tau = 0$, moving in the same direction as α. Show that the speed of the photo relative to α is e^τ. Does that contradict the speed of light being constant?

Exercise 7. Energy-momentum

Let α be a material particle of mass $m > 0$ in Minkowski spacetime. We recall that the *energy-momentum* of α is the vector $P = m\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}$.

Let o be a freely falling observer and let $\xi = (t, x, y, z)$ be an inertial coordinate system relative to o. One calls *energy of α relative to o E_o* the time component of P in the coordinate system ξ and *momentum of α relative to o* the space component $\vec{P}_o$ of P in the coordinate system ξ.

(1) Find the expressions of E_o and $\vec{P}_o$ in terms of the relative speed v. Show that for small values of v, E_o and $\vec{P}_o$ almost coincide with the Newtonian kinetic energy and momentum.

(2) The *scalar momentum* of α relative to o is $p_o \coloneqq \|\vec{P}_o\|^2$. Show that E, m, and p are related by $E^2 = m^2 + p^2$ and $E = pv$.

(3) Derive from the previous question that if a lightlike particle has a well-defined energy-momentum, then it should have zero mass. By considering a lightlike particle with energy E as the limit of a sequence of freely falling particles with constant energy E and increasing speeds, propose a definition for the energy-momentum of a lightlike particle.