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# General Relativity

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## Exercise Sheet 1 Exercise 5: Solution

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**Exercise 5. The Schwarzschild half-plane**

(1) If you don't want to memorize the formula for the Christoffel symbols of the metric, here is how to recover it. First we need to recover the Koszul formula, which gives the expression of the Levi-Civita connection. Start by writing the compatibility of the Levi-Civita connection with the metric:

$$X \cdot \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Use the torsion-freeness to rewrite the last term:

$$X \cdot \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle \tag{1}$$

Now write cyclic permutations of (1):

$$Y \cdot \langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle \tag{2}$$
$$Z \cdot \langle X, Y\rangle = \langle \nabla_Z Z, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle \tag{3}$$

And finally do (1) + (2) - (3):

$$X \cdot \langle Y, Z\rangle + Y \cdot \langle Z, X\rangle - Z \cdot \langle X, Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle Y, [X, Z]\rangle + \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle$$

So we recovered the Koszul formula:

$$2\langle \nabla_X Y, Z\rangle = X \cdot \langle Y, Z\rangle + Y \cdot \langle Z, X\rangle - Z \cdot \langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle - \langle X, [Z, Y]\rangle \tag{4}$$

Now let $(x^i)_{1\leqslant i\leqslant n}$ be a system of local coordinates, denote by $(e_i)$ the associated frame field (i.e. $e_i = \frac{\partial}{\partial x^i}$). Let's apply the Koszul formula (4) for $X = e_i$, $Y = e_j$, and $Z = e_l$. Recall that the coordinate vector fields always pair to zero with the Lie bracket, so there just remains:

$$2g(\Gamma_{ij}^m e_m, e_l) = e_i \cdot g(e_j, e_l) + e_j \cdot g(e_l, e_i) - e_l \cdot g(e_i, e_j) \tag{5}$$

(Recall that we always sum over repeated indices, so in (5) we sum over $m$.) We get:

$$2g_{ml}\Gamma_{ij}^m = g_{jl,i} + g_{li,j} - g_{ij,l}$$

where $g_{jl,i} := \frac{\partial g_{jl}}{\partial x_i}$, etc. We finally recover the formula for the Christoffel symbols by multiplying both sides by $g^{kl}$:

$$2\Gamma_{ij}^k = g^{kl}\left(g_{jl,i} + g_{li,j} - g_{ij,l}\right)$$

(Recall that $(g^{ij})$ is the inverse matrix of $(g_{ij})$, so $g^{kl}g_{ml} = \delta_m^k$).
Since $\Gamma_{ij}^k$ is symmetric in $i, j$, in the case of a surface ($n = 2$) there are only 6 Christoffel symbols to compute:

$$\begin{aligned}
2\Gamma_{11}^1 &= g^{11}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) + g^{12}\left(g_{12,1} + g_{21,1} - g_{11,2}\right)\\
2\Gamma_{11}^2 &= g^{21}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) + g^{22}\left(g_{12,1} + g_{21,1} - g_{11,2}\right)\\
2\Gamma_{12}^1 &= g^{11}\left(g_{21,1} + g_{11,2} - g_{12,1}\right) + g^{12}\left(g_{22,1} + g_{21,2} - g_{12,2}\right)\\
2\Gamma_{12}^2 &= g^{21}\left(g_{21,1} + g_{11,2} - g_{12,1}\right) + g^{22}\left(g_{22,1} + g_{21,2} - g_{12,2}\right)\\
2\Gamma_{22}^1 &= g^{11}\left(g_{21,2} + g_{12,2} - g_{22,1}\right) + g^{12}\left(g_{22,2} + g_{22,2} - g_{22,2}\right)\\
2\Gamma_{22}^2 &= g^{21}\left(g_{21,2} + g_{12,2} - g_{22,1}\right) + g^{22}\left(g_{22,2} + g_{22,2} - g_{22,2}\right)
\end{aligned}$$

If the coordinate system is orthogonal, i.e. the matrix of $g$ is diagonal, then there are quite a few simplifications:

$$\begin{aligned}
2\Gamma_{11}^1 &= g^{11}g_{11,1}\\
2\Gamma_{11}^2 &= -g^{22}g_{11,2}\\
2\Gamma_{12}^1 &= g^{11}g_{11,2}\\
2\Gamma_{12}^2 &= g^{22}g_{22,1}\\
2\Gamma_{22}^1 &= -g^{11}g_{22,1}\\
2\Gamma_{22}^2 &= g^{22}g_{22,2}
\end{aligned}$$

In the case at hand, we have the coordinates $(t, r)$, and coordinates of the metric tensor and its inverse given by:

$$\begin{aligned}
g_{11} &= -h & g^{11} &= -h^{-1}\\
g_{22} &= h^{-1} & g^{22} &= h
\end{aligned}$$

where $h(t, r) = h(r)$. The derivatives of the metric tensor are given by:

$$\begin{aligned}
g_{11,1} &= -\frac{\partial h}{\partial t} = 0\\
g_{11,2} &= -\frac{\partial h}{\partial r} = -h'\\
g_{22,1} &= \frac{\partial h^{-1}}{\partial t} = 0\\
g_{22,2} &= \frac{\partial h^{-1}}{\partial r} = -h'h^{-2}
\end{aligned}$$

So we find the Christoffel symbols:

$$\begin{aligned} \Gamma^1_{11} &= 0 \\ 2\Gamma^2_{11} &= hh' \\ 2\Gamma^1_{12} &= h^{-1}h' \\ \Gamma^2_{12} &= 0 \\ \Gamma^1_{22} &= 0 \\ 2\Gamma^2_{22} &= -h'h^{-1} \end{aligned}$$

(2) Since we are in dimension $n = 2$, there is only 1 sectional curvature (it is the Gaussian curvature of the surface):

$$K = \frac{\langle R(e_1, e_2)e_2, e_1\rangle}{\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle^2}$$

with:

$$\begin{aligned} R(e_1, e_2)e_2 &= \nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{e_2}\nabla_{e_1}e_2 \\ &= \nabla_{e_1}\left(\Gamma^1_{22}e_1 + \Gamma^2_{22}e_2\right) - \nabla_{e_2}\left(\Gamma^1_{12}e_1 + \Gamma^2_{12}e_2\right) \end{aligned}$$

In our case $\Gamma^1_{22} = \Gamma^2_{12} = 0$, so we're left with:

$$\begin{aligned} R(e_1, e_2)e_2 &= \nabla_{e_1}\left(\Gamma^2_{22}e_2\right) - \nabla_{e_2}\left(\Gamma^1_{12}e_1\right) \\ &= \Gamma^2_{22,1}e_2 + \Gamma^2_{22}\nabla_{e_1}e_2 - \Gamma^1_{12,2}e_1 - \Gamma^1_{12}\nabla_{e_2}e_1 \\ &= \Gamma^1_{22,1}e_2 + \Gamma^2_{22}\Gamma^1_{12}e_1 - \Gamma^1_{12,2}e_1 - \Gamma^1_{12}\Gamma^1_{21}e_1 \end{aligned}$$

where we used $\Gamma^2_{12} = \Gamma^2_{21} = 0$. This yields:

$$\langle R(e_1, e_2)e_2, e_1\rangle = \left(\Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12,2} - \Gamma^1_{12}\Gamma^1_{21}\right) g_{11}$$

In our case $\Gamma^2_{22} = -\Gamma^1_{12}$, so there remains:

$$\langle R(e_1, e_2)e_2, e_1\rangle = -\left(2(\Gamma^1_{12})^2 + \Gamma^1_{12,2}\right) g_{11}$$

which is:

$$\begin{aligned} \langle R(e_1, e_2)e_2, e_1\rangle &= \frac{-g_{11}}{2}\left((2\Gamma^1_{12})^2 + 2\Gamma^1_{12,2}\right) \\ &= \frac{h}{2}\left((h'h^{-1})^2 + (h'h^{-1})'\right) \\ &= \frac{h''}{2} \end{aligned}$$

On the other hand,

$$\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle^2 = g_{11}g_{22} = -1 \ .$$

So we find the expression of the sectional curvature:

$$K = \frac{-h''}{2} \ .$$

For $h(r) = 1 - \frac{r_S}{r}$, this is:

$$K = \frac{r_S}{r^3} \ .$$

(3) Let $\varphi(t, r) = (t + b, r)$. Then

$$\begin{aligned}
\varphi^* \, \mathrm{d}s^2 &= \varphi^*(-h \, \mathrm{d}t^2 + h^{-1} \, \mathrm{d}r^2) \\
&= -h(t + b, r) \, \mathrm{d}(t + b)^2 + h^{-1}(t + b, r) \, \mathrm{d}r^2 \\
&= -h \, \mathrm{d}t^2 + h^{-1} \, \mathrm{d}r^2 \\
&= \mathrm{d}s^2 \ .
\end{aligned}$$

We proved that $\varphi^* \, \mathrm{d}s^2 = \mathrm{d}s^2$, i.e. $\varphi$ is a isometric. It is also clearly a global diffeomorphism, so $\varphi$ is a global isometry. The proof works similarly for $\varphi(t, r) = (-t + b, r)$.

(4) It is predictable that the $t$-lines are geodesics, because they are fixed points of isometries $t \mapsto (-t + b, r)$.
Let us look for a function $\theta \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that the curve $\gamma(s) = (t = \theta(s), r_0)$ is a geodesic. The velocity of $\gamma$ is:

$$\begin{aligned}
\gamma'(s) &= \theta'(s) \frac{\partial}{\partial t} \\
&= \theta'(s) e_1 \ .
\end{aligned}$$

The "acceleration" of $\gamma$ is:

$$\begin{aligned}
\nabla_{\gamma'(s)} \gamma'(s) &= \nabla_{\theta'(s) e_1} (\theta'(s) e_1) \\
&= \theta'(s) \nabla e_1 (\theta'(s) e_1) \\
&= \theta'(s) \theta''(s) e_1 + (\theta'(s))^2 \nabla e_1 e_1 \\
&= \theta'(s) \theta''(s) e_1 + (\theta'(s))^2 \Gamma^2_{11}(\theta(s), r_0) e_2
\end{aligned}$$

where we used that $\Gamma^1_{22} = 0$.
Clearly, $\nabla_{\gamma'(s)} \gamma'(s) = 0$ if and only if $\theta'(s)\theta''(s) + (\theta'(s))^2 \Gamma^2_{11}(\theta(s), r_0) = 0$, i.e.:

$$\theta' \theta'' + (\theta')^2 h(r_0) h'(r_0) = 0 \ .$$

Note that $\theta' = 0$ is a solution, but that would mean that $\gamma$ is constant. It is not surprising that we find this solution but we must exclude it. If $\theta'$ is not identically 0, then it cannot vanish anywhere by uniqueness in the Cauchy-Lipschitz (Picard-Lindelöf) theorem. So we can divide by $\theta'$ and get:

$$\theta'' + h(r_0) h'(r_0) \theta' = 0 \ .$$

We can integrate:

$$\theta' + C_0\theta = C$$

where $C_0 = h(r_0)h'(r_0)$ and $C$ is any constant. This is linear first order ODE, it is easy to solve, e.g. with the method of integrating factors: multiply both sides by $\mu(s)$:

$$\mu(s)\,(\theta' + C_0\theta) = \mu(s)C \tag{6}$$

and choose $\mu(s)$ so that the left-hand side of (6) is $\frac{\mathrm{d}(\mu\theta)}{\mathrm{d}s}$. We take $\mu(s) = e^{C_0 s}$ and get:

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{C_0 s}\theta(s) = C e^{C_0 s}$$

and we can integrate again:

$$e^{C_0 s}\theta(s) = \frac{C}{C_0} e^{C_0 s} + C_1$$

where $C_1$ is another constant. Thus we find the solutions:

$$\theta(s) = C_2 + C_1 e^{-C_0 s}$$

where $C_2 = \frac{C}{C_0}$. So we find the geodesics:

$$\gamma(s) = \left(C_2 + C_1 e^{-C_0 s}, r_0\right) .$$

Note that the choice of $C_1 \in \mathbb{R}$ and $C_2 \in \mathbb{R} - \{0\}$ correspond to reparametrizations of the same geodesic. It is a time-like geodesic because its velocity is a multiple of $e_1 = \frac{\partial}{\partial_t}$, which is time-like since $g(e_1, e_1) = -h < 0$.

(5) Let us look for a function $\theta\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that the curve $\gamma(s) = (t_0, r = \theta(s))$ is a geodesic. The velocity of $\gamma$ is:

$$\begin{aligned} \gamma'(s) &= \theta'(s)\frac{\partial}{\partial r} \\ &= \theta'(s) e_2 \ . \end{aligned}$$

The "acceleration" of $\gamma$ is:

$$\begin{aligned} \nabla_{\gamma'(s)}\gamma'(s) &= \nabla_{\theta'(s)e_2}(\theta'(s)e_2) \\ &= \theta'(s)\nabla e_2(\theta'(s)e_2) \\ &= \theta'(s)\theta''(s)e_2 + (\theta'(s))^2\nabla e_2 e_2 \\ &= \theta'(s)\theta''(s)e_2 + (\theta'(s))^2\Gamma^2_{22}(t_0, \theta(s))e_2 \end{aligned}$$

where we used that $\Gamma^1_{22} = 0$.
Clearly, $\nabla_{\gamma'(s)}\gamma'(s) = 0$ if and only if $\theta'(s)\theta''(s) + (\theta'(s))^2\Gamma^2_{22}(t_0, \theta(s)) = 0$, i.e.:

$$\theta'\theta'' - (\theta')^2\frac{h'(\theta)}{h(\theta)} = 0 \ .$$

Note that $\theta' = 0$ is a solution, but that would mean that $\gamma$ is constant. It is not surprising that we find this solution but we must exclude it. If $\theta'$ is not identically 0, then it cannot vanish anywhere by uniqueness in the Cauchy-Lipschitz (Picard-Lindelöf) theorem. So we can divide by $\theta'$ and get:

$$\theta'' h(\theta) - \theta' h'(\theta) = 0 \ .$$

We thus find by integrating

$$\frac{\theta'}{h(\theta)} = C$$

where $C$ is a constant, which integrates as

$$F(\theta) = Cs + C_1$$

where $C_1$ is a constant and $F$ is an antiderivative of $h^{-1}$. Note that since $h^{-1} > 0$, $F$ is strictly monotonous, so it has a well-defined inverse, and we do find a unique solution

$$\gamma(s) = (t_0, F^{-1}(Cs + C_1))$$

up to the choice of $C$ and $C_1$, which corresponds to reparametrizations of the geodesic. In our case, we have $h^{-1}(r) = 1 + \frac{r_S}{r - r_S}$, so we can take $F(r) = r + r_S \ln(r - r_S)$. However the inverse function of $F$ is not easily accessible (one can express it in terms of the Lambert $W$-function, but it is not very instructive).

It is a space-like geodesic because its velocity is a multiple of $e_2 = \frac{\partial}{\partial_r}$, which is time-like since $g(e_2, e_2) = h^{-1} > 0$.

(6) The geodesic $s \mapsto \gamma(s)$ with this initial data exists and is unique, by uniqueness of geodesics with initial position and velocity. The initial velocity $\gamma'(0) = (1 + r_S)e_1 + e_2$ (at $\gamma(0) = (1, 1 + r_S)$) is lightlike:

$$\begin{aligned} g(\gamma'(0), \gamma'(0)) &= (1 + r_S)^2 g(e_1, e_1) + g(e_2, e_2) \\ &= -(1 + r_S)^2 \left(1 - \frac{r_S}{1 + r_S}\right) + \left(1 - \frac{r_S}{1 + r_S}\right)^{-1} \\ &= \left(1 - \frac{r_S}{1 + r_S}\right)^{-1} \left[1 - (1 + r_S)^2 \left(1 - \frac{r_S}{1 + r_S}\right)^2\right] \\ &= 0 \ . \end{aligned}$$

Since $\gamma'(0)$ is lightlike, the whole geodesic must be lightlike, since $g(\gamma'(s), \gamma'(s))$ is constant for any geodesic.

In order to find the explicit parametrization of the geodesic, we integrate the geodesic equation $\nabla_{\gamma'}\gamma' = 0$. Let us skip details, one finds:

$$\gamma(s) = (1 + s + r_S \ln(1 + s), 1 + s + r_S) \ .$$

(7) All the other lightlike geodesics are obtained from a single lightlike geodesic by applying isometries $(t, r) \mapsto (\pm t + b, r)$. Indeed, it is straightforward to check that such isometries act transitively on the lightlike tangent vectors up to scaling. Here is a picture taken from O'Neill p.153:

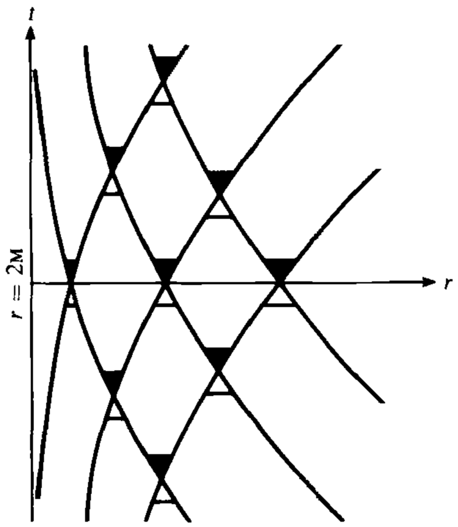


Figure 1: Lightlike geodesics in the Schwarzschild half-plane.