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General Relativity

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Exercise Sheet 1

Exercise 1. Hyperbolic space, de Sitter space, anti-de Sitter space

(1) Let (V, g) be a vector space with an inner product of index q, i.e. g has signature (p, q) with $p + q = \dim V$. Let $W \subseteq V$ be a subspace containing no nonzero null vectors. Show that the restriction of g to W is definite and that $V = W \oplus W^\perp$. What is the signature of g in restriction to $W^\perp$?

(2) Let $\mathbb{H}^n := \{v \in \mathbb{R}^{n+1}_1 \mid \langle v, v\rangle = -1 \text{ and } v_1 > 0\}$.
  (a) Show that $\mathbb{H}^n$ is a submanifold of $\mathbb{R}^{n+1}_1$. Show that $\mathrm{T}_v\mathbb{H}^n = \{v\}^\perp$.
  (b) Derive from (1) that $\mathbb{H}^n$ with the induced metric from $\mathbb{R}^{n+1}_1$ is a Riemannian manifold.
  (c) (*) Show that $\mathbb{H}^n$ has constant sectional curvature -1.

(3) Let $dS^n := \{v \in \mathbb{R}^{n+1}_1 \mid \langle v, v\rangle = 1\}$.
  (a) Following the same steps as (2), show that dS^n is a Lorentzian manifold.
  (b) (*) Show that dS^n has constant sectional curvature 1.

(4) Let $AdS^n := \{v \in \mathbb{R}^{n+1}_2 \mid \langle v, v\rangle = -1\}$.
  (a) Following the same steps as (2), show that AdS^n is a Lorentzian manifold.
  (b) (*) Show that AdS^n has constant sectional curvature 1.

Exercise 2. Isometries of Minkowski space

(1) We denote $O_1(n)$ the group of linear isometries of Minkowski space $\mathbb{R}^n_1$ (as a vector space with an inner product). This is called the *Lorentz group*. Can you describe the group $O_1(n)$ as a subgroup of $\mathrm{GL}(\mathbb{R}^n)$?

(2) (*) Show that the group of isometries of Minkowski space as an affine space, namely $\mathbb{R}^n \rtimes O_1(n)$, is equal to its group of isometries as a Riemannian manifold.

(3) We denote $O_1^+(n)$ the subgroup of $O_1(n)$ of isometries that preserve the direction of time (these are called *orthochronous*). Can you give a proper definition of $O_1^+(n)$?

(4) Let $c = 1$, $0 \leqslant v < c$, $\beta = \frac{v}{c}$, and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ (*Lorentz factor*). Show that the map $(t, x, y, z) \mapsto (t', x', y', z')$ with:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

is an orthochronous Lorentz transformation of $\mathbb{R}^4_1$. This is called a *Lorentz boost*.

Exercise 3. Time-orientability

(1) Recall the definition of time-orientability and time-orientation of a Lorentzian manifold N, in terms of timelike vector fields.

(2) By definition, two timelike vector fields X and Y give the same time-orientation of M if they define the same future- and past-pointing vectors. What is a condition for X and Y to give the same time-orientation?

(3) Let $\mathcal{T} := \{v \in \mathrm{T}M \mid v \neq 0 \text{ and } g(v,v) \leqslant 0\} \subseteq \mathrm{T}M$ denote the set of causal vectors (the *time cone*). Show that M is time-orientable if and only if $\mathcal{T}$ has two connected components.

Exercise 4. Constructing Lorentzian metrics

(1) Let (M,g) be a Riemannian manifold. Let U be a nonvanishing vector field on M and denote by $U^\flat$ the dual 1-form. Define $g_U := g - U^\flat \otimes U^\flat$. Show that (M, g_U) is a time-orientable Lorentzian manifold.

(2) Can you use (1) to construct a Lorentzian metric on S^2?

Exercise 5. The Schwarzschild half-plane

Let $r_S > 0$ be a constant ($r_S = \frac{2GM}{c^2}$ is the *Schwarzschild radius*). Define the *Schwarzschild half-plane*:

$$P = \{(t,r) \in \mathbb{R}^2 \mid r > r_S\}$$

with the metric

$$\mathrm{d}s^2 = -h\,\mathrm{d}t^2 + h^{-1}\,\mathrm{d}r^2$$

where

$$h(t,r) = h(r) = 1 - \frac{r_S}{r}\,.$$

(1) Compute the Christoffel symbols of the metric. *Do that computation for any $h = h(r)$.*

(2) Compute the sectional curvature. *Do that computation for any $h = h(r)$.*

(3) Prove that the maps $(t,r) \mapsto (\pm t + b, r)$ (b constant) are isometries.

(4) Show that the t-lines, i.e. curves $\{r = r_0\}$ with r_0 constant, can be parametrized as timelike geodesics.

(5) Show that the r-lines, i.e. curves $\{t = t_0\}$ with t_0 constant, can be parametrized as spacelike geodesics.

(6) Show that there exists a unique geodesic $(t(s), r(s))$ with $(t(0) = 1, r(0) = 1 + r_S)$ and $(t'(0) = 1 + r_S, r'(0) = 1)$, and that it is lightlike. Find an explicit parametrization of it.

(7) Explain how to find all light-like geodesics with (6) and (3). Draw a picture of the lightlike geodesics in the tr-plane.