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"He wants us to move the island."
John Locke

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I refer to the French version for more detailed and personal acknowledgments.

Résumé / Abstract

Résumé

Cette thèse est consacrée à l'étude de la géométrie symplectique complexe de l'espace de déformations des structures projectives complexes sur une surface. En explorant les connexions entre les différentes approches possibles de cette géométrie symplectique, l'auteur essaie d'en donner une description globale et unificatrice. La structure symplectique cotangente provenant de la paramétrisation schwarzienne est étudiée en détail et comparée à la structure symplectique canonique de la variété des caractères, clarifiant et généralisant un théorème de S. Kawai [Kaw96]. Il s'en ensuit une généralisation de résultats dûs à C. McMullen, notamment de la réciprocité quasifuchsienne. La structure symplectique cotangente est également abordée à travers la notion de surfaces minimales dans les variétés hyperboliques de dimension 3. Enfin, cette géométrie symplectique est décrite dans un cadre hamiltonien en relation avec les coordonnées de Fenchel-Nielsen complexes sur l'espace quasifuchsien, précisant les résultats obtenus par I. Platis [Pla01].

Mots-clefs

structure projective complexe, structure symplectique, théorie de Teichmüller, variété des caractères, géométrie hyperbolique, groupe kleinien, structure hyperkählerienne

The symplectic geometry of the deformation space of complex projective structures on a surface

Abstract

This thesis investigates the complex symplectic geometry of the deformation space of complex projective structures on a surface. The author attempts to give a global and unifying picture of this symplectic geometry by exploring the connections between different possible approaches. The cotangent symplectic structure given by the Schwarzian parametrization is studied in detail and compared to the canonical symplectic structure on the character variety, clarifying and generalizing a theorem of S. Kawai [Kaw96]. Generalizations of results of C. McMullen are derived, notably quasifuchsian reciprocity. The cotangent symplectic structure is also addressed through the notion of minimal surfaces in hyperbolic 3-manifolds. Finally, the symplectic geometry is described in a Hamiltonian setting with the complex Fenchel-Nielsen coordinates on the quasifuchsian space, recovering results of I. Platis [Pla01].

Keywords

complex projective structure, symplectic structure, Teichmüller theory, character variety, hyperbolic geometry, Kleinian group, hyperkähler structure

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Introduction

Complex projective structures on surfaces are examples of geometric structures. In the spirit of Klein's Erlangen program [1], a geometric structure is the structure of a locally homogeneous space. This notion, first initiated by Charles Ehresmann [Ehr36], embraces the celebrated Euclidean, spherical and hyperbolic geometries defined by homogeneous metrics. Complex projective structures on surfaces are very rich geometric structures in themselves. They include in particular the three classical Riemannian geometries previously mentioned and they extend the theory of complex structures on surfaces, *i.e.* Teichmüller theory.

The deformation space of geometric structures on a given manifold is a natural object of study, whose properties reflect features of the geometry of interest. Teichmüller theory is a fundamental example that has generated considerable attention, illuminating an immensely rich structure. One of the reasons for this is Teichmüller space's two-faceted complex vs hyperbolic nature. The complex approach is the more analytical of the two, using quasiconformal deformations and holomorphic quadratic differentials to develop the complex analysis of Teichmüller space. The hyperbolic approach is more geometric and naturally gives rise to the symplectic structure on Teichmüller space, hyperbolic length functions and twist deformations. While each approach is interesting in itself, it is the interplay between the two that is particularly fertile. For example, the complex and the symplectic structures on Teichmüller space combine into a Kähler structure. The study of complex projective structures on surfaces proves to be arguably even richer. First of all, as previously mentioned, projective structures comprise both hyperbolic and complex structures on surfaces. Secondly, they have an intimate connection with another class of geometric structures: hyperbolic structures on 3-manifolds. A third feature is their analytic description using the Schwarzian derivative, which turns the deformation space into

1. Felix Klein's 1872 *Erlangen program* [Kle93] attempted to define geometry as the study of a space that is invariant under a group of transformations. It is noteworthy that Klein emphasized projective geometry as the unifying frame for all geometries. The phenomenon of complex projective geometry of surfaces extending Euclidean, spherical, hyperbolic and complex geometries echoes this principle.

a holomorphic affine bundle modeled on the cotangent space to Teichmüller space. A natural complex symplectic geometry shows through these different perspectives, which has been discussed by various authors, e.g. [Kaw96], [Pla01] and [Gol04]. Just like the symplectic structure on Teichmüller space is the imaginary part of a Kähler structure, namely the Weil-Petersson metric, the complex symplectic structure on the deformation space of projective structures should appear as the "imaginary part" of a natural hyperkähler metric. However, this structure is poorly understood as of today. The initial motivation of the author's PhD was to investigate the hyperkähler structure of the deformation space of complex projective structures. Although no clear breakthroughs were achieved in that respect, some clarifications were discovered regarding the complex symplectic structure, in relation to key features of this deformation space. This thesis attempts a comprehensive and unifying picture of the complex symplectic geometry of the deformation space of complex projective structures on surfaces, one that carefully relates the different approaches [2]. It is the author's hope that this is not the end of the story, but rather a step towards the understanding of the hyperkähler geometry of the deformation space.

Let S be a closed surface of genus $g \geqslant 2$. A complex projective structure on S is given by an atlas of charts mapping open sets of S into the projective line $\mathbb{C}\mathbf{P}^1$ such that the transition maps are restrictions of Möbius transformations. The deformation space of projective structures $\mathcal{CP}(S)$ is the space of equivalence classes of projective structures on S, where two projective structures are considered equivalent if they are diffeomorphic [3]. Any projective atlas is in particular a holomorphic atlas, therefore a projective structure has an underlying complex structure. This gives a forgetful projection $p : \mathcal{CP}(S) \to \mathcal{T}(S)$, where $\mathcal{T}(S)$ is the Teichmüller space of S, defined as the deformation space of complex structures on S.

The Schwarzian derivative is a differential operator that turns the fibers of p into complex affine spaces. Globally, $\mathcal{CP}(S)$ is a holomorphic affine bundle modeled on the holomorphic cotangent bundle $T^*\mathcal{T}(S)$. This yields an identification $\mathcal{CP}(S) \approx T^*\mathcal{T}(S)$, but it is not canonical: it depends on the choice of the "zero section" $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$. There are at least two natural choices of sections to be considered. The Fuchsian section $\sigma_{\mathcal{F}}$ assigns to a Riemann surface X its Fuchsian projective structure given by the uniformization theorem. However, $\sigma_{\mathcal{F}}$ is not

2. Of course, this has already been done at least partially by authors including the three previously mentioned.

3. More precisely, diffeomorphic by a homotopically trivial diffeomorphism, see section 1.1.

holomorphic. The other natural choice is that of a Bers section, given by Bers' simultaneous uniformization theorem. Bers sections are a family of holomorphic sections parametered by Teichmüller space. Like any holomorphic cotangent bundle, $T^*\mathcal{T}(S)$ is equipped with a canonical complex symplectic form ω, *i.e.* a nondegenerate closed $(2,0)$-form. Each choice of a zero section σ thus yields a symplectic structure ω^σ on $\mathcal{CP}(S)$, simply by pulling back the canonical symplectic form on $T^*\mathcal{T}(S)$. A first natural question is: How is ω^σ affected by σ? This is answered in section 2.3:

Proposition 2.3. For any two sections σ_1 and σ_2 to $p : \mathcal{CP}(S) \to \mathcal{T}(S)$,

$$\omega^{\sigma_2} - \omega^{\sigma_1} = -p^* d(\sigma_2 - \sigma_1)\ . \tag{1}$$

A significantly different description of $\mathcal{CP}(S)$ is given by the holonomy of complex projective structures. The holonomy is a concept defined for any geometric structures, in this situation it gives a local identification $hol : \mathcal{CP}(S) \to \mathcal{X}(S, PSL_2(\mathbb{C}))$, where the character variety $\mathcal{X}(S, PSL_2(\mathbb{C}))$ is defined as a quotient of the set of representations $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$. By a general construction of Goldman, $\mathcal{X}(S, PSL_2(\mathbb{C}))$ enjoys a natural complex symplectic structure ω_G. How is this symplectic structure related to the cotangent symplectic structures ω^σ introduced above? A theorem of Kawai [Kaw96] gives a pleasant answer to that question: If σ is any Bers section, then ω^σ and ω_G [4] agree up to some constant. Kawai's proof is highly technical and not very insightful though. Also, the conventions chosen in his paper can be misleading [5]. Relying on theorems of other authors, we give a simple alternative proof of Kawai's result. In fact, we are able to do a little better and completely answer the question raised above. Our argument is based on the observation that there is an intricate circle of related ideas:

(**i**) $p : \mathcal{CP}(S) \to \mathcal{T}(S)$ is a Lagrangian fibration (with respect to ω_G).
(**ii**) Bers sections $\mathcal{T}(S) \to \mathcal{CP}(S)$ are Lagrangian (with respect to ω_G).
(**iii**) If M is a quasifuchsian 3-manifold, the map $\beta : \mathcal{T}(\partial_\infty M) \to \mathcal{CP}(\partial_\infty M)$ [6] is Lagrangian (with respect to ω_G).
(**iv**) ω_G restricts to the Weil-Petersson Kähler form ω_{WP} on the Fuchsian slice.
(**v**) If σ is any Bers section, then $d(\sigma_{\mathcal{F}} - \sigma) = -i\omega_{WP}$.

4. We mean here $hol^* \omega_G$ rather than ω_G, but we abusively use the same notation for the two (as explained in section 3.3).

5. With the conventions chosen in his paper, Kawai finds $\omega^\sigma = \pi\omega_G$. Compare with our result: $\omega^\sigma = -i\omega_G$. Whether the constant is real or imaginary does matter when taking the real and imaginary parts, obviously, and this can be significant (in particular when trying to figure out the Kähler forms that generate hyperkähler structure). Kawai's choices imply that ω_G takes imaginary values in restriction to the Fuchsian slice, which does not seem very relevant. Goldman showed in [Gol84] that (with appropriate conventions) ω_G is just the Weil-Petersson Kähler form on the Fuchsian slice. For the interested reader, we believe that, even after rectifying the conventions, there is a factor 2 missing in Kawai's result.

6. β is given by Bers' simultaneous uniformization theorem, see section 1.4

(**vi**) McMullen's quasifuchsian reciprocity (see [McM00] and Theorem 5.18).

(**vii**) For any Bers section σ, $\omega^\sigma = -i\omega_G$.

Let us briefly comment on these. (**iv**) is a result due to Goldman ([Gol84]). (**v**) and (**vi**) are closely related and due to McMullen ([McM00]). Steven Kerckhoff discovered that (**iii**) easily follows from a standard argument, exposed in e.g. [KS], and we include this argument in our presentation (Theorem 3.3) for completeness. (**vii**) appears to be the strongest result, as it is not too hard to see that it implies all other results [7]. However, using Proposition 2.3 written above and a simple analytic continuation argument (Theorem 5.7), we show that (**iv**) and (**v**) imply (**vii**). In fact, we give a characterization of sections σ such that ω^σ agrees with ω:

Theorem 5.8. Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be a section to p. Then ω^σ agrees with the standard complex symplectic structure ω_G on $\mathcal{CP}(S)$ if and only if $\sigma_{\mathcal{F}} - \sigma$ is a primitive for the Weil-Petersson metric on $\mathcal{T}(S)$:

$$\omega^\sigma = \omega_G \iff d(\sigma_{\mathcal{F}} - \sigma) = \omega_{WP} \ . \tag{2}$$

(**vii**) then follows from McMullen's theorem (**v**):

Theorem 5.10. If $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ is any Bers section, then

$$\omega^\sigma = -i\omega_G \ . \tag{3}$$

We also get the expression of the symplectic structure pulled back by the Fuchsian identification:

Corollary 5.13. Let $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S)$ be the Fuchsian section. Then

$$\omega^{\sigma_{\mathcal{F}}} = -i(\omega_G - p^*\omega_{WP}) \ . \tag{4}$$

Generalizing these ideas in the setting of convex cocompact 3-manifolds, we define the notion of generalized Bers sections (see section 1.5) and prove a generalized version of Theorem 5.10, relying on a result of Takhtajan-Teo [TT03] [8]:

Theorem 5.15. Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section . Then

$$\omega^\sigma = -i\omega_G \ . \tag{5}$$

We derive a generalization of McMullen's result (**v**):

Corollary 5.17. Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section. Then

$$d(\sigma_{\mathcal{F}} - \sigma) = -i\omega_{WP} \ . \tag{6}$$

7. This is not entirely true *per se*, but we do not want to go into too much detail here.
8. However, we stress that the proof also relies indirectly on Theorem 5.10.

and a generalized version of McMullen's quasifuchsian reciprocity:

Theorem 5.18. Let $f : \mathcal{T}(S_j) \to \mathcal{CP}(S_k)$ and $g : \mathcal{T}(S_k) \to \mathcal{CP}(S_j)$ be reciprocal generalized Bers embeddings. Then $D_{X_j} f$ and $D_{X_k} g$ are dual maps. In other words, for any $\mu \in T_{X_j}\mathcal{T}(S_j)$ and $\nu \in T_{X_k}\mathcal{T}(S_k)$,

$$\langle D_{X_j} f(\mu), \nu \rangle = \langle \mu, D_{X_k} g(\nu) \rangle \ . \tag{7}$$

Next we discuss the "minimal surface identification". Other than the affine isomorphisms $\tau^\sigma : \mathcal{CP}(S) \xrightarrow{\sim} T^*\mathcal{T}(S)$ that depend on the choice of sections σ, there is another identification of interest between $\mathcal{CP}(S)$ and $T^*\mathcal{T}(S)$, although it is only defined in a neighborhood $\mathcal{AF}(S)$ on the Fuchsian slice $\mathcal{F}(S) = \sigma_{\mathcal{F}}(\mathcal{T}(S))$. This identification arises through the notion of minimal surfaces in hyperbolic 3-manifolds, and it should prove relevant in the investigation of the hyperkähler structure on $\mathcal{CP}(S)$. If $\Sigma \subset M$ is a minimal surface in a hyperbolic 3-manifold M, an observation going back to Hopf ([Hop51]) shows that the second fundamental form of Σ is the real part of a unique holomorphic quadratic differential on Σ, thus relating to the Teichmüller tangent covectors. Riemannian arguments that will not be developed here show that any quasifuchsian 3-manifold M associated to a projective structure Z that is near the Fuchsian slice $\mathcal{F}(S)$ contains a unique minimal surface Σ, which is diffeomorphic to S. This defines a map $\alpha : \mathcal{AF}(S) \to T^*\mathcal{T}(S)$, where $\alpha(Z) =: \varphi$ is the holomorphic quadratic differential such that $\mathrm{Re}(\varphi) = II_\Sigma$. It follows from the "fundamental theorem of surface theory" that α is a diffeomorphism onto some neighborhood of the zero section in $T^*\mathcal{T}(S)$. Again, it is legitimate to try to compare the symplectic structure ω_G on $\mathcal{CP}(S)$ and the canonical symplectic structure ω on $T^*\mathcal{T}(S)$ through the identification α. The mathematical setting now has a very "Riemannian" flavour, and it is necessary to involve new tools and ideas that are more adapted to this setting. The renormalized volume of hyperbolic manifolds is such a tool, one that proves crucial in comparing the symplectic structures. This is a function $W : \mathcal{AF}(S) \to \mathbb{R}$ defined using a "renormalization" that gives a finite notion of volume to otherwise volume-infinite subsets of hyperbolic manifolds, and it is related to the notion of equidistant foliations of hyperbolic ends. All of these ideas are made precise in sections 2.4 and 5.4. Borrowing arguments mainly from [KS08] we prove the following theorem:

Theorem 5.28. Let $W : \mathcal{AF}(S) \to \mathbb{R}$ be the renormalized volume function on the almost-Fuchsian space, α be the minimal surface identification as in section 2.4. Then

$$dW = -\frac{1}{4}\mathrm{Re}\left[\alpha^*\xi + (\tau^{\sigma_{\mathcal{F}}})^*\xi\right] \tag{8}$$

where ξ is the canonical one-form on the complex cotangent space $T^*\mathcal{T}(S)$ (see section 2.2).

A direct consequence of this and Corollary 5.13 mentioned above is the following identification of real symplectic structures on $\mathcal{CP}(S)$:

Corollary 5.29.

$$\mathrm{Re}(\alpha^*\omega) = -\mathrm{Im}(\omega_G) \ . \tag{9}$$

Finally, we discuss the symplectic geometry of $\mathcal{CP}(S)$ in relation to complex Fenchel-Nielsen coordinates on the quasifuchsian space $\mathcal{QF}(S)$. These are global holomorphic coordinates on $\mathcal{QF}(S)$ introduced by Kourouniotis ([Kou94]) and Tan ([Tan94]) that are the complexification of the classical Fenchel-Nielsen coordinates on Teichmüller space $\mathcal{T}(S)$, or rather the Fricke space $\mathcal{F}(S)$. In [Wol82], [Wol83] and [Wol85], Wolpert showed that the Fenchel-Nielsen coordinates on $\mathcal{F}(S)$ are intimately related to the symplectic structure. For any simple closed curve γ on the surface S, there is a hyperbolic length function $l_\gamma : \mathcal{F}(S) \to \mathbb{R}$ and a twist flow $\mathrm{tw}_\gamma : \mathbb{R} \times \mathcal{F}(S) \to \mathcal{F}(S)$. Given a pants decomposition $\alpha = (\alpha_1, \dots, \alpha_N)$ [9] on S, choosing a section to $l_\alpha = (l_{\alpha_1}, \dots, l_{\alpha_N})$ yields the classical Fenchel-Nielsen coordinates on Teichmüller space $(l_\alpha, \tau_\alpha) : \mathcal{F}(S) \to (\mathbb{R}_{>0})^N \times \mathbb{R}^N$. Wolpert showed that the twist flow associated to a curve γ is the Hamiltonian flow of the length function l_γ. He also gave formulas for the Poisson bracket of two length functions, which show in particular that the length functions l_{α_i} associated to a pants decomposition α define an integrable Hamiltonian system, for which the functions l_{α_i} are the action variables and the twist functions τ_{α_i} are the angle variables. In [Pla01], Platis shows that this very nice "Hamiltonian picture" remains true in its complexified version on the quasifuchsian space for some complex symplectic structure ω_P, giving complex versions of Wolpert's results. This Hamiltonian picture is also extensively explored on the $SL_2(\mathbb{C})$-character variety by Goldman in [Gol04]. Independently from Platis' work, our analytic continuation argument shows that complex Fenchel-Nielsen coordinates are Darboux coordinates for the symplectic structure on $\mathcal{QF}(S)$:

Theorem 5.19. Let α be any pants decomposition of S. Complex Fenchel-Nielsen coordinates $(l_\alpha^{\mathbb{C}}, \beta_\alpha^{\mathbb{C}})$ on the quasifuchsian space $\mathcal{QF}(S)$ are Darboux coordinates for the standard complex symplectic structure:

$$\omega_G = \sum_{i=1}^{N} dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}} \tag{10}$$

and this shows in particular that

Corollary 5.20. Platis' symplectic structure ω_P is equal to the standard complex symplectic structure ω_G on the quasifuchsian space $\mathcal{QF}(S)$ [10].

9. *i.e.* a maximal collection of nontrivial distinct free homotopy classes of simple closed curves, see section 4.1.

10. This fact is mentioned as "apparent" in [Pla01] and is somewhat implied in [Gol04], but it would seem to the author that it was not formally proved.

We thus recover Platis' and some of Goldman's results, in particular that the complex twist flow is the Hamiltonian flow of the associated complex length function. Although in the Fuchsian case it would seem unnecessarily sophisticated to use this as a definition of the twist flow, this approach might be fruitful in the space of projective structures. This transformation relates to what other authors have called *quakebends* or *complex earthquakes* discussed by Epstein-Marden [EM87], McMullen [McM98], Series [Ser01] among others.

Outline of the thesis:

- **Chapter 1** introduces Teichmüller space $\mathcal{T}(S)$ and the deformation space of complex projective structures $\mathcal{CP}(S)$. Fuchsian and quasifuchsian structures are discussed, as well as the connection between projective structures and 3-dimensional hyperbolic structures.
- **Chapter 2** explains the identifications between $\mathcal{CP}(S)$ and the cotangent space $T^*\mathcal{T}(S)$ and examines the cotangent symplectic structures obtained as a result.
- **Chapter 3** discusses the character variety $\mathcal{X}(S, PSL_2(\mathbb{C}))$, the holonomy of complex projective structures and Goldman's symplectic structure.
- **Chapter 4** presents the complex Fenchel-Nielsen coordinates on the quasi-fuchsian space and their Hamiltonian description.
- **Chapter 5** carries through the exploration of the connections between the different symplectic structures and the ramifications involved. This chapter contains most of our results.
- **Chapter 6** presents hyperkähler structures and outlines some perspectives for future research.

CHAPTER 1

Teichmüller space and the deformation space of complex projective structures

1.1 Definition of $\mathcal{T}(S)$ and $\mathcal{CP}(S)$

Let S be a surface. Unless otherwise stated, we will assume that S is connected, oriented, smooth, closed and with genus $g \geqslant 2$ [1].

A complex structure on S is a maximal atlas of charts mapping open sets of S into the complex line $\mathbb{C}$ such that the transition maps are holomorphic transformations. The atlas is required to be compatible with the orientation and smooth structure on S. A Riemann surface X is a surface S equipped with a complex structure.

The group $\mathrm{Diff}^+(S)$ of orientation preserving diffeomorphisms of S acts on the set of all complex structures on S in a natural way: a compatible complex atlas on S is pulled back to another one by such diffeomorphisms. Denote by $\mathrm{Diff}^+_0(S)$ the identity component of $\mathrm{Diff}^+(S)$, its elements are the orientation preserving diffeomorphisms of S that are homotopic to the identity. The quotient $\mathcal{T}(S)$ of the set of all complex structures on S by $\mathrm{Diff}^+_0(S)$ is called the *Teichmüller space* of S, its elements are called *marked Riemann surfaces*.

In a similar fashion, define a *complex projective structure* on S as a maximal atlas of charts mapping open sets of S into the complex projective line $\mathbb{C}\mathbf{P}^1$ such that the transition maps are projective transformations (restrictions of fractional linear transformations). The atlas is also required to be compatible with the orientation and smooth structure on S. A complex projective surface Z is a surface S equipped with

1. In some sections (e.g. 1.5), we will allow S to be disconnected in order to be able to consider the case where S is the boundary of a compact 3-manifold. This does not cause any problem in the exposition above.

a complex projective structure. In terms of geometric structures (see e.g. [Thu97]), a complex projective structure is a $(\mathbb{C}\mathbf{P}^1, PSL_2(\mathbb{C}))$-structure.

Again, $\mathrm{Diff}^+(S)$ naturally acts on the set of all complex projective structures on S. The quotient $\mathcal{CP}(S)$ by the subgroup $\mathrm{Diff}_0^+(S)$ is called the *deformation space of complex projective structures* on S, its elements are *marked complex projective surfaces.*

1.2 $\mathcal{T}(S)$ and $\mathcal{CP}(S)$ are complex manifolds

Kodaira-Spencer deformation theory (see [KS58], also [EE69]) applies and it shows that $\mathcal{T}(S)$ is naturally a complex manifold with holomorphic tangent space $T_X^{1,0}\mathcal{T}(S) = H^1(X, \Theta_X)$, where Θ_X is the sheaf of holomorphic vector fields on X. Denote by K the canonical bundle over X (the holomorphic cotangent bundle of X). By Dolbeault's theorem, $H^1(X, \Theta_X)$ is isomorphic to the Dolbeault cohomology space $H^{-1,1}(X)$. Elements of $H^{-1,1}(X)$ are Dolbeault classes of smooth sections of $K^{-1} \otimes \bar{K}$, which are called *Beltrami differentials.* In a complex chart $z : U \subset S \to \mathbb{C}$, a Beltrami differential μ has an expression of the form $\mu = u(z)\frac{d\bar{z}}{dz}$ where u is a smooth function. The fact that we only consider (Dolbeault) classes of Beltrami differentials can be expressed as follows : if V is a vector field on X of type $(1,0)$, then the Beltrami differential $\overline{\partial} V$ induces a trivial (infinitesimal) deformation of the complex structure X. Recall that X carries a unique hyperbolic metric within its conformal class (called the Poincaré metric) by the uniformization theorem. By Hodge theory, every Dolbeault cohomology class has a unique harmonic representative μ. The tangent space $T_X\mathcal{T}(S)$ is thus also identified with the space $HB(X)$ of harmonic Beltrami differentials.

We can also derive a nice description of the Teichmüller cotangent space using (co)homology machinery. $H^1(X, \Theta_X) = H^1(X, K^{-1})$ because $\dim_{\mathbb{C}} X = 1$ and this space is dual to $H^0(X, K^2)$ by Serre duality. An element $\varphi \in Q(X) := H^0(X, K^2)$ is called a *holomorphic quadratic differential.* In a complex chart $z : U \subset S \to \mathbb{C}$, φ has an expression of the form $\varphi = \phi(z)dz^2$, where ϕ is a holomorphic function. The holomorphic cotangent space $T_X^*\mathcal{T}(S)$ is thus identified with the space $Q(X)$ of holomorphic quadratic differentials. The duality pairing $Q(X) \times H^{-1,1}(X) \to \mathbb{C}$ is just given by $(\varphi, \mu) \mapsto \int_S \varphi \cdot \mu$. Note that we systematically use tensor contraction (when dealing with line bundles over X) : $\varphi \cdot \mu$ is a section of $K \otimes \bar{K} \approx |K|^2$, so it defines a conformal density and can be integrated over S. With the notations above, $\varphi \cdot \mu$ has local expression $\phi(z)u(z)|dz|^2$.

An easy consequence of the Riemann-Roch theorem is that $\dim_{\mathbb{C}} Q(X) = 3g - 3$, so that $\mathcal{T}(S)$ is a complex manifold of dimension $3g - 3$.

Similarly, Kodaira-Spencer deformation theory applies to show that $\mathcal{CP}(S)$ is naturally a complex manifold with tangent space $T_Z\mathcal{CP}(S) = H^1(Z, \Xi_Z)$, where Ξ_Z is the sheaf of projective vector fields on Z (see also [Hub81]). It follows that $\mathcal{CP}(S)$ is a complex manifold of dimension $6g - 6$.

Unlike Teichmüller tangent vectors, there is no immediate way to describe tangent vectors to $\mathcal{CP}(S)$ in a more tangible way. However, note that a complex projective atlas is in particular a holomorphic atlas, so that a complex projective surface Z has an underlying structure of a Riemann surface X. This yields a forgetful map

$$p : \mathcal{CP}(S) \to \mathcal{T}(S) \tag{1.1}$$

which is easily shown to be holomorphic. We will see in section 2.1 that the fiber $p^{-1}(X)$ is naturally a complex affine space whose underlying vector space is $Q(X)$. In particular $\dim_{\mathbb{C}} \mathcal{CP}(S) = \dim_{\mathbb{C}} \mathcal{T}(S) \times \dim_{\mathbb{C}} Q(X) = 6g - 6$ as expected.

1.3 The Weil-Petersson Kähler metric on $\mathcal{T}(S)$

The *Weil-Petersson product* of two holomorphic quadratic differentials Φ and Ψ is given by

$$\langle \varphi, \psi \rangle_{WP} = -\frac{1}{4} \int_X \varphi \cdot \sigma^{-1} \cdot \overline{\psi} \tag{1.2}$$

where σ^{-1} is the dual current of the area form σ for the Poincaré metric. It is a Hermitian product on the complex vector space $Q(X)$. In a complex chart with values in the upper half-plane $z = x + iy : U \subset X \to \mathbb{H}^2$, the area form is given by

$$\sigma = \frac{dx \wedge dy}{y^2} = \frac{-2i dz \wedge d\overline{z}}{(z - \overline{z})^2} \ ,$$

its dual current is

$$\sigma^{-1} = y^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = \frac{i}{2}(z - \overline{z})^2 dz^{-1} \wedge d\overline{z}^{-1} \ .$$

The tensor product $-\frac{1}{4}\varphi \cdot \sigma^{-1} \cdot \overline{\psi}$ reduces to the classical expression

$$\frac{-i}{8}(z - \overline{z})^2 \varphi(z)\overline{\psi(z)} dz \wedge d\overline{z} = y^2 \varphi(z)\overline{\psi(z)} dx \wedge dy \ .$$

By duality, this gives a Hermitian product also denoted by $\langle \cdot, \cdot \rangle_{WP}$ on $H^{-1,1}(X)$ and globally a Hermitian metric $\langle \cdot, \cdot \rangle_{WP}$ on the manifold $\mathcal{T}(S)$. It was first shown to be Kähler by Ahlfors [Ahl61] and Weil.

The Kähler form of the Weil-Petersson metric on $\mathcal{T}(S)$ is the real symplectic form

$$\omega_{WP} = -\mathrm{Im} \langle \cdot, \cdot \rangle_{WP} \ . \tag{1.3}$$

1.4 Fuchsian and quasifuchsian structures

Note that whenever a Kleinian group Γ (*i.e.* a discrete subgroup of $PSL_2(\mathbb{C})$) acts freely and properly on some open subset U of the complex projective line $\mathbb{C}\mathbf{P}^1$, the quotient surface U/Γ inherits a complex projective structure. This gives a variety (but not all) of complex projective surfaces, called embedded projective structures (see section 3.3).

In particular, if S is equipped with a marked complex structure X, the uniformization theorem provides S with a marked complex projective structure as follows. The uniformization theorem gives a representation $\rho : \pi_1(S) \to PSL_2(\mathbb{R})$ such that $X \approx \mathbb{H}^2/\Gamma$ as Riemann surfaces, where $\mathbb{H}^2$ is the upper half-plane and $\Gamma := \rho(\pi_1(S))$. $\mathbb{H}^2$ can be seen as an open set (a disk) in $\mathbb{C}\mathbf{P}^1$ and a Fuchsian group $\Gamma \subset PSL_2(\mathbb{R})$ is in particular a Kleinian group, so the quotient $X \approx \mathbb{H}^2/\Gamma$ inherits a complex projective structure Z. Moreover, this structure is compatible with the complex structure X (in the sense that $p(Z) = X$). In other words, this defines a section

$$\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S) \tag{1.4}$$

to p, called the *Fuchsian section*. This shows in particular that the projection p is surjective. We call $\mathcal{F}(S) := \sigma_{\mathcal{F}}(\mathcal{T}(S))$ the (deformation) space of (standard) Fuchsian structures on S, it is an embedded copy of $\mathcal{T}(S)$ in $\mathcal{CP}(S)$.

Quasifuchsian structures are another useful example of embedded projective structures. Let us quickly recall how such structures are defined. Given two marked complex structures $(X^+, X^-) \in \mathcal{T}(S)\times\mathcal{T}(\overline{S})$ [2] (where $\overline{S}$ is the surface S with reversed orientation), Bers' simultaneous uniformization theorem states that there exists a unique representation $\rho : \pi_1(S) \xrightarrow{\sim} \Gamma \subset PSL_2(\mathbb{C})$ up to conjugation such that:

- The limit set [3] Λ is a Jordan curve. The domain of discontinuity Ω is then the disjoint union of two simply connected domains Ω^+ and Ω^-. Such a Γ is called a quasifuchsian group.
- As marked Riemann surfaces, $X^+ \approx \Omega^+/\Gamma$ and $X^- \approx \Omega^-/\Gamma$.

Again, both Riemann surfaces X^+ and X^- inherit complex projective structures Z^+ and Z^- by this construction. This defines a map

$$\beta = (\beta^+, \beta^-) : \left\{ \begin{array}{ccc} \mathcal{T}(S) \times \mathcal{T}(\overline{S}) & \to & \mathcal{CP}(S) \times \mathcal{CP}(\overline{S}) \\ (X^+, X^-) & \mapsto & (\beta^+(X^+, X^-), \beta^-(X^+, X^-)) \end{array} \right. \tag{1.5}$$

2. Note that $\mathcal{T}(\overline{S})$ is canonically identified with $\overline{\mathcal{T}(S)}$, which denotes the manifold $\mathcal{T}(S)$ equipped with the opposite complex structure. The same remark holds for $\mathcal{CP}(\overline{S})$ and $\overline{\mathcal{CP}(S)}$.

3. The *limit set* Λ is defined as the complement in $\mathbb{C}\mathbf{P}^1$ of the domain of discontinuity Ω, which is the maximal open set on which Γ acts freely and properly. Alternatively, Λ is described as the closure in $\mathbb{C}\mathbf{P}^1$ of the set of fixed points of elements of Γ.

which is a holomorphic section to $p \times p : \mathcal{CP}(S) \times \mathcal{CP}(\overline{S}) \to \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ by Bers' theorem. The map β has the obvious symmetry property: $\overline{\beta^-(X^+, X^-)} = \beta^+(\overline{X^-}, \overline{X^+})$.

In particular, when $X^- \in \mathcal{T}(\overline{S})$ is fixed, the map

$$\sigma_{X^-} := \beta^+(\cdot, X^-) : \mathcal{T}(S) \to \mathcal{CP}(S) \tag{1.6}$$

is a holomorphic section to p, called a *Bers section*, and its image $\sigma_{X^-}(\mathcal{T}(S))$ in $\mathcal{CP}(S)$ will be called a *Bers slice.* On the other hand, when $X^+ \in \mathcal{T}(S)$ is fixed, the map

$$f_{X^+} = \beta^+(X^+, \cdot) \tag{1.7}$$

is an embedding of $\mathcal{T}(\overline{S})$ in the fiber $P(X^+) = p^{-1}(X^+) \subset \mathcal{CP}(S)$ [4], f_{X^+} is called a *Bers embedding.* Also, note that $\sigma_{\mathcal{F}}(X) = \beta^+(X, \bar{X}) = \overline{\beta^-(X, \bar{X})} = \sigma_{\bar{X}}(X)$. This shows that the Fuchsian section $\sigma_{\mathcal{F}}$ is real analytic but not holomorphic, in fact it is a maximal totally real analytic embedding, see section 5.1.

We will call $\mathcal{QF}(S) := \beta^+(\mathcal{T}(S) \times \mathcal{T}(\overline{S})) \subset \mathcal{CP}(S)$ the (deformation) space of (standard) quasifuchsian structures on S. It is an open neighborhood of $\mathcal{F}(S)$ in $\mathcal{CP}(S)$ (this is a consequence of general arguments mentioned in the next paragraph), and it follows from the discussion above that Bers slices and Bers embeddings define two transverse foliations of $\mathcal{QF}(S)$ by holomorphic copies of $\mathcal{T}(S)$.

1.5 Complex projective structures and hyperbolic 3-manifolds

In this section, we briefly review the relation between complex projective structures on the boundary of a compact 3-manifold $\hat{M}$ and hyperbolic structures on its interior. References for this section include [Kap09] and [CM04]. The quasifuchsian picture presented in the previous section occurs as a particular case of this discussion. We then define *generalized Bers sections* and *generalized Bers embeddings*, and fix a few notations that will be useful later on.

Let M be a connected complete hyperbolic 3-manifold. The universal cover of M is isometric to hyperbolic 3-space $\mathbb{H}^3$, this defines a unique faithful representation $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3) \approx PSL_2(\mathbb{C})$ (up to conjugation) such that $\Gamma := \rho(\pi_1(M))$ acts freely and properly on $\mathbb{H}^3$ and $M \approx \mathbb{H}^3/\Gamma$. Let $\Omega \subset \mathbb{C}\mathbf{P}^1$ be the domain of discontinuity of the Kleinian group Γ, it is the maximal open set on which Γ acts freely and properly. Here $\mathbb{C}\mathbf{P}^1$ is seen as the "ideal boundary" of $\mathbb{H}^3$, also denoted $\partial_\infty \mathbb{H}^3$. The possibly disconnected surface $\partial_\infty M := \Omega/\Gamma$ is called the *ideal boundary* of M and it inherits a complex projective structure as the quotient of $\Omega \subset \mathbb{C}\mathbf{P}^1$ by the Kleinian group Γ. Conversely, any torsion-free Kleinian group Γ acts freely and

4. which has the structure of a complex affine space as we will see in section 2.1.

properly on $\mathbb{H}^3 \sqcup \Omega$ (where Ω is the domain of discontinuity of Γ), and the quotient consists of a 3-manifold $\hat{M} = M \sqcup \partial_\infty M$, where $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold and $\partial_\infty M = \Omega/\Gamma$ is its ideal boundary. In general, the manifold $\hat{M}$ is not compact, if it is then $\hat{M}$ is topologically the end compactification of M. In that case we say that the hyperbolic structure on M is *convex cocompact*. Let us finally define the *convex core* of M (which will be used later): it is the quotient of the convex hull of the limit set Λ in $\mathbb{H}^3$ by Γ. It is well-known that M is convex cocompact if and only if its convex core is a compact deformation retract of M.

Consider now a smooth 3-manifold with boundary $\hat{M}$ with the following topological restrictions: $\hat{M}$ is connected, oriented, compact, irreducible [5], atoroidal [6] and with infinite fundamental group. Let $M = \hat{M} \setminus \partial\hat{M}$ denote the interior of $\hat{M}$. For simplicity, we also assume that the boundary $\partial\hat{M}$ is incompressible [7] and contains no tori, so that it consists of a finite number of surfaces $S_1, \ldots, S_N$ of genera at least 2. The Teichmüller space $\mathcal{T}(\partial\hat{M})$ is described as the direct product

$$\mathcal{T}(\partial\hat{M}) = \mathcal{T}(S_1) \times \cdots \times \mathcal{T}(S_N)\ , \tag{1.8}$$

similarly

$$\mathcal{CP}(\partial\hat{M}) = \mathcal{CP}(S_1) \times \cdots \times \mathcal{CP}(S_N)\ , \tag{1.9}$$

and there is a holomorphic "forgetful" projection

$$p = p_1 \times \cdots \times p_N : \mathcal{CP}(\partial\hat{M}) \to \mathcal{T}(\partial\hat{M})\ . \tag{1.10}$$

We also let $\mathrm{pr}_k : \mathcal{CP}(\partial\hat{M}) \to \mathcal{CP}(S_k)$ denote the k^{th} projection map. Let us consider the space $\mathcal{HC}(M)$ of convex cocompact hyperbolic structures on M up to homotopy. In other words, we define $\mathcal{HC}(M)$ as the quotient of the set of convex cocompact hyperbolic metrics on M by the group of orientation-preserving diffeomorphisms of M that are homotopic to the identity. Let us mention that Marden [Mar74] and Sullivan [Sul85] showed that $\mathcal{HC}(M)$ is a connected component of the interior of the subset of discrete and faithful representations in the character variety $\mathcal{X}(M, PSL_2(\mathbb{C}))$. By the discussion above, any element of $\mathcal{HC}(M)$ determines a marked complex projective structure $Z \in \mathcal{CP}(\partial\hat{M})$. We thus have a map $\varphi : \mathcal{HC}(M) \to \mathcal{CP}(\partial\hat{M})$, and it is shown to be holomorphic, this is a straightforward consequence of the fact that the holonomy map is holomorphic (see section 3.3). Considering the induced conformal structure on $\partial\hat{M}$, define the map $\psi = p \circ \varphi$ as in the following diagram:

$$\begin{tikzcd} \mathcal{HC}(M) \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & \mathcal{CP}(\partial\hat{M}) \arrow[d, "p"] \\ & \mathcal{T}(\partial\hat{M})\ . \end{tikzcd}$$

5. meaning that every embedded 2-sphere bounds a ball.
6. meaning that it does not contain any embedded, non-boundary parallel, incompressible tori.
7. meaning that the map $\iota_* : \pi_1(\partial\hat{M}) \to \pi_1(\hat{M})$ induced by the inclusion map ι is injective.

The powerful theorem mainly due to Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurson[8] says in this context that:

Theorem 1.1 (Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurson)**.** *The map $\psi : \mathcal{HC}(M) \to \mathcal{T}(\partial\hat{M})$ is bijective.*

Let us mention that this statement has to be slightly modified if $\hat{M}$ has compressible boundary. As a direct consequence of this theorem, we get

Proposition 1.2. *The map*

$$\beta = \varphi \circ \psi^{-1} : \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M}) \tag{1.11}$$

is a canonical holomorphic section to $p : \mathcal{CP}(\partial\hat{M}) \to \mathcal{T}(\partial\hat{M})$.

We call β the *(generalized) simultaneous uniformization section.* This map allows us to define "generalized Bers sections" and "generalized Bers embeddings" by letting only one of the boundary components' conformal structure vary and by looking at the resulting complex projective structure on some other (or the same) boundary component. This idea is precised in the following. If an index $j \in \{1, \dots, N\}$ and marked complex structures $X_i \in \mathcal{T}(S_i)$ are fixed for all $i \neq j$, we denote by $\iota_{(X_i)}$ the canonical injection

$$\iota_{(X_i)} : \begin{array}{ccc} \mathcal{T}(S_j) & \to & \mathcal{T}(\partial\hat{M}) \\ X & \mapsto & (X_1, \dots, X_{j-1}, X, X_{j+1}, \dots, X_N)\ . \end{array} \tag{1.12}$$

Let $f_{(X_i),k} = \mathrm{pr}_k \circ \beta \circ \iota_{(X_i)}$ as in the following diagram:

$$\begin{tikzcd} \mathcal{T}(\partial\hat{M}) \arrow[r, "\beta"] & \mathcal{CP}(\partial\hat{M}) \arrow[d, "\mathrm{pr}_k"] \\ \mathcal{T}(S_j) \arrow[u, "\iota_{(X_i)}"] \arrow[r, "f_{(X_i),k}"] & \mathcal{CP}(S_k)\ . \end{tikzcd} \tag{1.13}$$

If $j = k$, then $\sigma_{(X_i)} := f_{(X_i),j}$ is a holomorphic section to $p_j : \mathcal{CP}(S_j) \to \mathcal{T}(S_j)$, that we call a *generalized Bers section.* On the other hand, if $j \neq k$, then $f_{(X_i),k}$ maps $\mathcal{T}(S_j)$ in the affine[9] fiber $P(X_k) \subset \mathcal{CP}(S_k)$, we call a $f_{(X_i),k}$ a *generalized Bers embedding.* We apologize for this ambiguous terminology: a "generalized Bers embedding" is *not* an embedding in general.

8. see [CM04] chapter 7. for a detailed exposition of this theorem, containing in particular the description of the different contributions of the several authors. A non-exhaustive list of references includes [AB60], [Ahl64], [Ber87], [Kra72], [Mar74], [Mas71], [Sul85].

9. see section 2.1.

Note that quasifuchsian structures discussed in the previous paragraph just correspond to the case where $M = S \times \mathbb{R}$. Let us also mention that this discussion is easily adapted when ∂M contains tori or is no longer assumed incompressible, with a few precautions. When $\hat{M}$ only has one boundary component, this gives the notion of a *Schottky section*.

CHAPTER 2

The cotangent symplectic structure

2.1 $\mathcal{CP}(S)$ as an affine holomorphic bundle over $\mathcal{T}(S)$

The Schwarzian derivative

Given a locally injective holomorphic function $f : Z_1 \to Z_2$ where Z_1 and Z_2 are complex projective surfaces, define the *osculating map* $\tilde{f}$ to f at a point $m \in Z_1$ as the germ of a (locally defined) projective map that has the best possible contact with f at m. In some sense, one can take a flat covariant derivative $\nabla \tilde{f}$ and identify it as holomorphic quadratic differential $\mathcal{S}f \in Q(X)$, called the *Schwarzian derivative* of f. We refer to [And98] and [Dum09] for details.

In local projective charts, the Schwarzian derivative of f has the classical expression $\mathcal{S}f = Sf(z)dz^2$, where

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 .$$

As a consequence of the definition, the Schwarzian operator enjoys the following properties:

Proposition 2.1.

- *If f is a projective map, then $\mathcal{S}f = 0$ (the converse is also true).*
- *If $f : Z_1 \to Z_2$ and $g : Z_2 \to Z_3$ are locally injective holomorphic functions between complex projective surfaces, then*

$$\mathcal{S}(g \circ f) = \mathcal{S}(f) + f^* \mathcal{S}(g) .$$

The Schwarzian derivative also satisfies an existence theorem:

Proposition 2.2. *If $U \subset \mathbb{C}$ is simply connected and $\varphi \in Q(U)$, then $\mathcal{S}f = \varphi$ can be solved for $f : U \to \mathbb{C}\mathbf{P}^1$.*

An elementary and constructive proof of this fact is given in e.g. [Dum09], see also [And98] for a more abstract argument.

Schwarzian parametrization of a fiber

Recall that there is a holomorphic "forgetful" map $p : \mathcal{CP}(S) \to \mathcal{T}(S)$. Let X be a fixed point in $\mathcal{T}(S)$ and $P(X) := p^{-1}(\{X\})$ the set of marked projective structures on S whose underlying complex structure is X.

Given Z_1, $Z_2 \in P(X)$, the identity map $\mathrm{id}_S : Z_1 \to Z_2$ is holomorphic but not projective if $Z_1 \neq Z_2$. Taking its Schwarzian derivative accurately measures the "difference" of the two projective structures Z_1 and Z_2. Let us make this observation more precise. A consequence of Proposition 2.2 is that given $Z_1 \in P(X)$ and $\varphi \in Q(X)$, there exists $Z_2 \in P(X)$ such that $\mathcal{S}\left(\mathrm{id}_S : Z_1 \to Z_2\right) = \varphi$. This defines a map $Q(X) \times P(X) \to P(X)$, which is now easily seen to be a freely transitive action of $Q(X)$ on $P(X)$ as a consequence of Proposition 2.1. In other words, $P(X)$ is equipped with a complex affine structure, modeled on the vector space $Q(X)$.

Recall that $Q(X)$ is also identified with the complex dual space $T^*_X\mathcal{T}(S)$, so that globally $\mathcal{CP}(S)$ is an affine holomorphic bundle modeled on the holomorphic cotangent vector bundle $T^*\mathcal{T}(S)$.

As a consequence, we can identify $\mathcal{CP}(S)$ with $T^*\mathcal{T}(S)$ by choosing a section to p which serves the purpose of the zero section. Explicitely, if σ is a smooth section to p, we get an identification $\tau^\sigma : Z \mapsto Z - \sigma\left(p(Z)\right)$ of smooth complex affine bundles as in the following diagram:

$$\begin{tikzcd} \mathcal{CP}(S) \arrow[rr, "\tau^\sigma"] \arrow[dr, "p"'] & & T^*\mathcal{T}(S) \arrow[dl, "\pi"] \\ & \mathcal{T}(S) \,. & \end{tikzcd} \tag{2.1}$$

τ^σ is characterized by the fact that $\tau^\sigma \circ \sigma$ is the zero section to $\pi : T^*\mathcal{T}(S) \to \mathcal{T}(S)$. It is an isomorphism of holomorphic bundles whenever σ is a holomorphic section to p, such as a Bers section or a generalized Bers section (see sections 1.4 and 1.5).

2.2 Complex symplectic structure on $T^*\mathcal{T}(S)$

It is a basic fact that if M is any complex manifold (in particular when $M = \mathcal{T}(S)$), the total space of its holomorphic cotangent bundle T^*M [1] is equipped with a canonical complex symplectic structure. We briefly recall this and a few useful properties.

The *canonical* 1*-form* ξ is the holomorphic $(1,0)$-form on T^*M defined at a point $\varphi \in T^*M$ by $\xi_\varphi := \pi^*\varphi$, where $\pi : T^*M \to M$ is the canonical projection and φ is seen as a complex covector on M in the right-hand side of the equality. The *canonical complex symplectic form* on T^*M is then simply defined by $\omega = d\xi$ [2]. If (z_k) is a system of holomorphic coordinates on M so that an arbitrary $(1,0)$-form has an expression of the form $\alpha = \sum w_k dz_k$, then (z_k, w_k) is a system of holomorphic coordinates on T^*M for which $\xi = \sum w_k dz_k$ and $\omega = \sum dw_k \wedge dz_k$.

The canonical 1-form satisfies the following reproducing property. If α is any $(1,0)$-form on M, it is in particular a map $M \to T^*M$ and as such it can be used to pull back differential forms from T^*M to M. It is then not hard to show that

$$\alpha^*\xi = \alpha \tag{2.2}$$

and as a consequence

$$\alpha^*\omega = d\alpha\ . \tag{2.3}$$

Note that if u is a vertical tangent vector to T^*M, *i.e.* $\pi_* u = 0$, then u can be identified with an element of the fiber containing its basepoint α (since the fibers of the projection are vector spaces). Under that identification, for any other tangent vector β, the symplectic pairing of u and v is just given by

$$\omega(u, v) = \langle u, \pi_* v\rangle\ , \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing on T_xM (here $x = \pi(\alpha)$).

Note that the fibers of the projection $\pi : T^*M \to M$ are Lagrangian submanifolds of T^*M, in other words π is a Lagrangian fibration. Also, the zero section $s_0 : M \hookrightarrow T^*M$ is a Lagrangian embedding. These are direct consequences of the previous observation.

2.3 The affine identification

As we have seen in section 2.1, any choice of a "zero section" $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ yields an affine isomorphism $\tau^\sigma : \mathcal{CP}(S) \xrightarrow{\sim} T^*\mathcal{T}(S)$. We can use this to pull back

1. In this context T^*M stands for the complex dual of the holomorphic tangent bundle $T^{1,0}M$, its (smooth) sections are the $(1,0)$-forms.

2. Note that some authors might take the opposite sign convention for ω.

the canonical symplectic structure of $T^*\mathcal{T}(S)$ on $\mathcal{CP}(S)$: define

$$\omega^\sigma := (\tau^\sigma)^*\omega \ . \tag{2.5}$$

It is clear that ω^σ is a complex symplectic form on $\mathcal{CP}(S)$ whenever σ is a holomorphic section to p. Otherwise, it is just a complex-valued nondegenerate 2-form on $\mathcal{CP}(S)$, whose real and imaginary parts are both real symplectic forms.

How is ω^σ affected by the choice of the "zero section" σ? The following statement is both straightforward and key:

Proposition 2.3. *For any two sections σ_1 and σ_2 to $p : \mathcal{CP}(S) \to \mathcal{T}(S)$,*

$$\omega^{\sigma_2} - \omega^{\sigma_1} = -p^* d(\sigma_2 - \sigma_1) \tag{2.6}$$

where $\sigma_2 - \sigma_1$ is the "affine difference" between σ_2 and σ_1, it is a 1-form on $\mathcal{T}(S)$. In particular, the symplectic structures induced by the respective choices of two sections agree if and only if their affine difference is a closed 1-form.

Proof. This is an easy computation:

$$\begin{aligned} -p^*d\,(\sigma_2 - \sigma_1) &= -p^*\left((\sigma_2 - \sigma_1)^*\omega\right) \quad \text{(see (2.3))} \\ &= \left(-(\sigma_2 - \sigma_1)\circ p\right)^*\omega \\ &= \left(\tau_2^\sigma - \tau_1^\sigma\right)^*\omega \\ &= \left(\tau^{\sigma_2}\right)^*\omega - \left(\tau^{\sigma_1}\right)^*\omega \ . \end{aligned}$$

Only the last step is not so trivial as it would seem because one has to be careful about basepoints. Also, note that in the identity

$$\tau^{\sigma_2}(Z) - \tau^{\sigma_1}(Z) = (Z - \sigma_2 \circ p(Z)) - (Z - \sigma_1 \circ p(Z)) = -(\sigma_2 - \sigma_1)\circ p(Z) \ ,$$

some minus signs are "affine" ones (hiding the Schwarzian derivative) and others are "real" minus signs, but this can be ignored in computation. □

Moreover, a straightforward calculation gives an explicit expression of $\omega^\sigma(u, v)$ whenever u is a vertical tangent vector to $\mathcal{CP}(S)$, it is exactly the same as the one obtained for the symplectic structure on $T^*\mathcal{T}(S)$:

Proposition 2.4. *Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be a section to p. Let Z be a point in $\mathcal{CP}(S)$, and u, v be tangent vectors at Z such that u is vertical, i.e. $p_*u = 0$. Then*

$$\omega^\sigma(u, v) = \langle u, p_*v\rangle \ . \tag{2.7}$$

In this expression, u is seen as an element of $\in T_X{}^*T(S)$ (where $X = p(Z)$) under the identification $T_Z P(X) = Q(X) = T_X{}^*T(S)$. Note that this expression not involving σ is compatible with the previous proposition, which implies that $\omega^{\sigma_2} - \omega^{\sigma_1}$ is a horizontal 2-form.

As a consequence, just like in the cotangent space, we have:

Proposition 2.5. *Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be any section. The projection $p : \mathcal{CP}(S) \to \mathcal{T}(S)$ is a Lagrangian fibration for ω^σ. Also, σ is a Lagrangian embedding.*

2.4 The minimal surface identification

There is another identification of interest between $\mathcal{CP}(S)$ and $T^*\mathcal{T}(S)$, though it is not globally defined, using the theory of minimal surfaces in hyperbolic 3-manifolds. First let us recall a few basic facts about this theory. We refer to [KS07] for details.

Extrinsic invariants of surfaces in hyperbolic 3-manifolds and the Gauss-Codazzi equations

Consider an immersed surface Σ in an oriented hyperbolic 3-manifold M. Denote by g the hyperbolic metric on M and $\bar{\nabla}$ its Levi-Civita connection. The following are classical extrinsic invariants of Σ:

- The *first fundamental form* I is the Riemannian metric on Σ induced by the hyperbolic metric g. Let ∇ denote its Levi-Civita connection and K its curvature.
- The *shape operator* $B \in \mathrm{End}(T\Sigma)$ is defined by $Bv := -\bar{\nabla}_v n$, where n is the positively oriented unit normal vector field to Σ. It is self-adjoint with respect to I. The *mean curvature* is defined by $H := \mathrm{tr}(B)$.
- The *second fundamental form* II is the symmetric bilinear form associated to B with respect to I: $II(u, v) := I(Bu, v) = I(u, Bv)$. We use the following notation convention: $B = I^{-1}\, II$.
- The *third fundamental form* III is the symmetric bilinear form defined by $III(u, v) = I(Bu, Bv)$.

These satisfy the *Gauss-Codazzi equations* on Σ:

$$\begin{cases} \det B = K + 1 & \text{(Gauss equation)} \\ d^\nabla B = 0 & \text{(Codazzi equation)} \end{cases} \tag{2.8}$$

where B is seen as a $T\Sigma$-valued one-form in the Codazzi equation and d^∇ is the extension of the exterior derivative using the connection ∇.

Conversely, the "fundamental theorem of surface theory" states that if I is a Riemannian metric on a surface Σ and $I\!I$ is a symmetric bilinear form on $T\Sigma$ such that I and $I\!I$ satisfy the Gauss-Codazzi equations, then there is an (essentially unique) immersion of Σ in a possibly non-complete hyperbolic 3-manifold M such that I and $I\!I$ are the first and second fundamental forms of Σ.

Minimal surfaces in hyperbolic 3-manifolds and holomorphic quadratic differentials

A *minimal surface* in a hyperbolic 3-manifold M is a minimally isometrically immersed Riemannian surface (Σ, I) in M. By the "fundamental theorem of surface theory", this is equivalent to the three following conditions on the extrinsic invariants of Σ:

$$\begin{cases} \det B = K + 1 \\ d^{\nabla} B = 0 \\ H = 0 \ . \end{cases} \tag{2.9}$$

The following lemma was first discovered by Hopf and is quite straightforward to prove but it provides a surprising relation between minimal surfaces and holomorphic quadratic differentials:

Lemma 2.6 ([Hop51]). *Let Σ be an oriented surface equipped with a Riemannian metric I and a symmetric bilinear form $I\!I$ on $T\Sigma$. Let $B := I^{-1} I\!I$. Consider the conformal class $[I]$, so that $X := (\Sigma, [I])$ is a Riemann surface.*

(i) $I\!I$ is the real part of a (unique) smooth quadratic differential φ if and only if $\operatorname{tr}(B) = 0$.

(ii) If (i) holds, then φ is holomorphic on X if and only if $d^{\nabla} B = 0$.

In particular, any embedded minimal surface $\Sigma \subset M$ in a hyperbolic 3-manifold defines a Riemann surface $X := (\Sigma, [I])$ and a holomorphic quadratic differential $\varphi \in Q(X)$, *i.e.* a point in the holomorphic cotangent of the Teichmüller space of Σ (see section 1.2).

Almost Fuchsian structures and the minimal surface identification

Let $M = M_Z$ denote the hyperbolic 3-manifold associated to a quasifuchsian projective structure $Z \in \mathcal{QF}(S)$ (see section 1.4). It is easy to see that if Z is a Fuchsian structure, then there is a unique minimal embedded surface $\Sigma \subset M$. In fact, the convex core of the Fuchsian manifold M is reduced to a totally geodesic surface, since the limit set of a Fuchsian representation is a circle in $\mathbb{C}\mathbf{P}^1$, and the minimal surface Σ is precisely that surface. Arguments exposed in e.g. [KS07] show that if Z stays in some neighborhood $\mathcal{AF}(S)$ of the Fuchsian slice $\mathcal{F}(S)$, there is still a unique minimal embedded surface $\Sigma \subset M_Z$ (still located inside the convex core of M). $\mathcal{AF}(S)$ is called the *(deformation) space of almost-Fuchsian structures* on S. Moreover, the normal exponential map provides an isotopic deformation of Σ on the ideal boundary component $\partial^+_\infty M$ and in particular a marking of the underlying smooth surface $\Sigma \approx S$.

By the previous discussion, each almost Fuchsian structure Z defines a point in $T^*\mathcal{T}(S)$: we get a map

$$\alpha : \begin{array}{ccc} \mathcal{AF}(S) & \to & T^*\mathcal{T}(S) \\ Z & \mapsto & ([I_\Sigma], \varphi) \end{array} \tag{2.10}$$

where Σ is the minimal surface in M_Z and $\varphi \in Q(X)$ is the holomorphic quadratic differential such that $II_\Sigma = \mathrm{Re}(\varphi)$.

We call α the *minimal surface identification*. On the Fuchsian slice $\mathcal{F}(S)$, α restricts to the identification of $\mathcal{F}(S)$ with the zero section of $T^*\mathcal{T}(S)$, just like the Fuchsian identification $\tau^{\sigma_\mathcal{F}}$ (see (2.1)). But contrary to the affine identifications τ^σ, α is not a bundle homomorphism (the conformal structure on Σ does not agree with that of Z in general). Also, α is not holomorphic with respect to the standard complex structures on $\mathcal{CP}(S)$ and $T^*\mathcal{T}(S)$. It is however an embedding of $\mathcal{AF}(S)$ in some neighborhood of the zero section of $T^*\mathcal{T}(S)$, this is a consequence of the "fundamental theorem of surface theory".

With this identification, we get two real symplectic structures on $\mathcal{AF}(S)$, namely the real and imaginary parts of $\alpha^*\omega$.

CHAPTER 3

The character variety and Goldman's symplectic structure

3.1 The character variety

References for this section include [Gol84], [HP04], [Gol04] and [Dum09].

Let $G := PSL_2(\mathbb{C})$ and $\mathcal{R}(S) := \mathcal{R}(S, G)$ be the set of group homomorphisms (representations) from $\pi := \pi_1(S)$ to G. It has a natural structure of a complex affine algebraic set as follows. Choose a finite presentation $\pi = \langle \gamma_1, \ldots, \gamma_N \mid (r_i)_{i \in I} \rangle$ of π. Evaluating a representation $\rho \in \mathcal{R}(S)$ on the generators γ_k embeds $\mathcal{R}(\mathcal{S})$ as an algebraic subset of G^N. This gives $\mathcal{R}(S)$ an affine structure indeed because of the identification $PSL_2(\mathbb{C}) \approx SO_3(\mathbb{C})$ (given by the adjoint representation of $PSL_2(\mathbb{C})$ on its Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$). It is easy to check that this structure is independent of the presentation.

G acts algebraically on $\mathcal{R}(S)$ by conjugation. The character variety $\mathcal{X}(S)$ is defined as the quotient in the sense of invariant theory. Specifically, the action of G on $\mathcal{R}(S)$ induces an action on the ring of regular functions $\mathbb{C}[\mathcal{R}(S)]$. Denote by $\mathbb{C}[\mathcal{R}(S)]^G$ the ring of invariant functions, it is finitely generated because $\mathcal{R}(S)$ is affine and G is reductive.

Lemma 3.1 (see e.g. [HP04])**.** *In fact, it is generated in this case ($G = PSL_2(\mathbb{C})$) by a finite number of the complex valued functions on $\mathcal{R}(S)$ of the form $\rho \mapsto \mathrm{tr}^2(\rho(\gamma))$.*

$\mathcal{X}(S)$ is the affine set such that $\mathbb{C}[\mathcal{X}(S)] = \mathbb{C}[\mathcal{R}(S)]^G$, it is called the *character variety* of S. A consequence of the lemma is that the points of $\mathcal{X}(S)$ are in one-to-one correspondance with the set of *characters*, *i.e.* complex-valued functions of the form $\gamma \in \pi \mapsto \mathrm{tr}^2(\rho(\gamma))$.

The affine set $\mathcal{X}(S)$ splits into two irreducible components $\mathcal{X}(S)_l \cup \mathcal{X}(S)_r$, where elements of $\mathcal{X}(S)_l$ are characters of representations that lift to $SL(2,\mathbb{C})$.

The set-theoretic quotient $\mathcal{R}(S)/G$ is rather complicated, but G acts freely and properly on the subset $\mathcal{R}(S)^s$ of irreducible [1] ("stable") representations, so that the quotient $\mathcal{R}(S)^s/G$ is a complex manifold. Furthermore, an irreducible representation is determined by its character, so that $\mathcal{X}(S)^s := \mathcal{R}(S)^s/G$ embeds (as a Zariski-dense open subset) in the smooth locus of $\mathcal{X}(S)$. Its dimension is $6g-6$. Let us mention that more generally, $\mathcal{X}(S)$ is in bijection with the set of orbits of "semistable" (*i.e.* reductive [2]) representations.

It is relatively easy to see that the Zariski tangent space at a point $\rho \in \mathcal{R}(S)$ is described as the space of crossed homomorphisms $Z^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$ (*i.e.* 1-cocycles in the sense of group cohomology), specifically maps $u : \pi \to \mathfrak{sl}_2(\mathbb{C})$ such that $u(\gamma_1\gamma_2) = u(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)} u(\gamma_2)$ [3]. The subspace corresponding to the tangent space of the G-orbit of ρ is the space of principal crossed homomorphisms $B^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$ (*i.e.* 1-coboundaries in the sense of group cohomology), specifically maps $u : \pi \to \mathfrak{sl}_2(\mathbb{C})$ such that $u(\gamma) = \mathrm{Ad}_{\rho(\gamma)} u_0 - u_0$ for some $u_0 \in \mathfrak{sl}_2(\mathbb{C})$. Hence for (at least) smooth points $[\rho] \in \mathcal{X}(S)$, the tangent space is given by $T_{[\rho]}\mathcal{X}(S) = H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$.

3.2 The complex symplectic structure on the character variety

By the general construction of [Gol84], the character variety enjoys a complex symplectic structure defined in this situation as follows.

Recall that the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is equipped with its complex Killing form B . It is a nondegenerate complex bilinear symmetric form preserved by G under the adjoint action. Let $\tilde{B} = \frac{1}{4}B$, it is explicitely given by $\tilde{B}(u,v) = \mathrm{tr}(uv)$ where $u, v \in \mathfrak{sl}_2(\mathbb{C})$ are represented by trace-free 2×2 matrices.

One can compose the standard cup-product in group cohomology with $\tilde{B}$ [4] as "coefficient pairing" to get a dual pairing

$$H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \times H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \xrightarrow{\cup} H^2(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho} \otimes \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \xrightarrow{\tilde{B}} H^2(\pi, \mathbb{C}) \cong \mathbb{C}\ . \tag{3.1}$$

This pairing defines a nondegenerate complex bilinear alternate 2-form on

1. A representation $\rho : \pi \to PSL_2(\mathbb{C})$ is called *irreducible* if it fixes no point in $\mathbb{C}\mathbf{P}^1$.
2. A nontrivial representation $\rho : \pi \to PSL_2(\mathbb{C})$ is called *reductive* if it is either irreducible of it fixes a pair of distinct points in $\mathbb{C}\mathbf{P}^1$.
3. where of course $\mathrm{Ad} : G \to \mathrm{Aut}\mathfrak{g}$ is the adjoint representation.
4. It would look somewhat more natural to use the actual Killing form B instead of $\tilde{B} = \frac{1}{4}B$, but we choose to go with $\tilde{B}$ because it is the convention used by most authors. Moreover, it gives a slightly simpler expression of our theorems 5.10, 5.15 and 5.19.

$H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \approx T_{[\rho]}\mathcal{X}(S)$. It globalizes into a nondegenerate 2-form ω_G on $\mathcal{X}(S)^s$. By arguments of Goldman ([Gol84]) following Atiyah-Bott ([AB83]) this form is closed, in other words it is a complex symplectic form on the smooth quasi-affine variety $\mathcal{X}(S)^s$ [5].

3.3 Holonomy of projective structures

Just like any geometric structure, a complex projective structure Z defines a *developing map* and a *holonomy representation* (see e.g. [Thu97]). The developing map is a locally injective projective map $f : \tilde{Z} \to \mathbb{C}\mathbf{P}^1$ and it is equivariant with respect to the holonomy representation $\rho : \pi \to PSL_2(\mathbb{C})$ in the sense that $f \circ \gamma = \rho(\gamma) \circ f$ for any $\gamma \in \pi$. When f is an embedding, Z is called an *embedded projective structure*. It is obtained as the quotient U/Γ, where U is the image of f and Γ is the image of ρ. Fuchsian and quasifuchsian structures are examples of embedded projective structure.

Holonomy defines a map

$$hol : \mathcal{CP}(S) \to \mathcal{X}(S) \ .$$

It is differentiable and its differential is "the identity map" in the sense that it is the canonical identification

$$d\,hol : T_Z\mathcal{CP}(S) = H^1(Z, \Xi_Z) \xrightarrow{\sim} H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) = T_{[hol(Z)]}\mathcal{X}(S) \ .$$

A consequence of this observation is that *hol* is a local biholomorphism.

The holonomy representation ρ of a complex projective structure satisfies the following properties:

- ρ is liftable to $SL_2(\mathbb{C})$ (a lift is provided by the monodromy of the Schwarzian equation). The image of the holonomy map thus lies in the irreducible component $\mathcal{X}(S)_l$ of $\mathcal{X}(S)$.
- The action of $\Gamma := \rho(\pi)$ on hyperbolic 3-space $\mathbb{H}^3$ does not fix any point or ideal point, nor does it preserve any geodesic. Representations having this property are called non-elementary. They are in particular irreducible representations, hence smooth points of the character variety as expected.

Conversely, it has been shown ([GKM00]) that any liftable non-elementary representation is the holonomy of a complex projective structure.

Although the holonomy map $hol : \mathcal{CP}(S) \to \mathcal{X}(S)$ is a local biholomorphism, it is neither injective nor a covering onto its image ([Hej75]). Nonetheless, we get a complex symplectic structure on $\mathcal{CP}(S)$ simply by pulling back that of $\mathcal{X}(S)^s$ by the

5. In fact, it defines an algebraic tensor on the whole character variety, see [Gol84].

holonomy map. Abusing notations, we will still call this symplectic structure ω_G. Alternatively, one could directly define ω_G on $\mathcal{CP}(S)$ in terms of the exterior product of 1-forms with values in some flat bundle (recall that $T_Z\mathcal{CP}(S) = H^1(Z, \Xi_Z)$, where Ξ_Z is the sheaf of projective vector fields on Z, see section 1.2). *We will consider ω_G as the "standard" complex symplectic structure on $\mathcal{CP}(S)$* (notably because it does not depend on any choice).

3.4 Fuchsian holonomy and Goldman's theorem

Let $\mathcal{F}(S)$ be the space of marked hyperbolic structures on S (we abusively use the same notation as for the Fuchsian space). More precisely, $\mathcal{F}(S)$ is the space of hyperbolic metrics on S quotiented by $\mathrm{Diff}_0^+(S)$. In terms of geometric structures, $\mathcal{F}(S)$ is the deformation space of $(\mathbb{H}^2, PSL_2(\mathbb{R}))$-structures on S (this is a consequence of Cartan-Hadamard's theorem). It follows that the holonomy map identifies $\mathcal{F}(S)$ as the connected component of the character variety $\mathcal{X}(S, PSL_2(\mathbb{R}))$ corresponding to faithful and discrete representations. $\mathcal{F}(S)$ is sometimes called the Fricke space of S.

The uniformization theorem states that there is a unique hyperbolic metric in each conformal class of Riemannian metrics on S. Since S is oriented, the choice of a conformal structure on S is equivalent to that of a complex structure on S. The uniformization theorem thus provides a bijective map

$$u : \mathcal{T}(S) \to \mathcal{F}(S) \ .$$

By definition of the Fuchsian section $\sigma_{\mathcal{F}}$, the map u is precisely identified as $\sigma_{\mathcal{F}}$ if hyperbolic structures are considered as particular cases of complex projective structures. Putting it differently, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{CP}(S) \arrow[rr, "hol"] & & \mathcal{X}(S, PSL_2(\mathbb{C})) \\
\mathcal{T}(S) \arrow[u, "\sigma_{\mathcal{F}}"] \arrow[r, "u"] & \mathcal{F}(S) \arrow[r, hook] & \mathcal{X}(S, PSL_2(\mathbb{R})) \ . \arrow[u, "\iota"']
\end{tikzcd} \tag{3.2}$$

It can easily be derived from this diagram that $\sigma_{\mathcal{F}}$ is a maximal totally real [6] analytic embedding of $\mathcal{T}(S)$ in $\mathcal{CP}(S)$.

By Goldman's general construction in [Gol84] described above in the case of $G = PSL_2(\mathbb{C})$ but which also works for $G = PSL_2(\mathbb{R})$ [7], $\mathcal{X}(S, PSL_2(\mathbb{R}))$ is equipped with a real symplectic structure $\omega_{G,PSL_2(\mathbb{R})}$. Of course it is just the restriction of the

6. see section 5.1 for a definition of this notion and a different argument for this fact.
7. In fact, for any reductive Lie group.

symplectic structure $\omega_G = \omega_{G,PSL_2(\mathbb{C})}$ on $\mathcal{X}(S, PSL_2(\mathbb{R})$. Recall that $\mathcal{T}(S)$ is also equipped with a symplectic structure, the Weil-Petersson Kähler form ω_{WP}. In the same article, Goldman shows the following identification (with our conventions):

Theorem 3.2 (Goldman [Gol84])**.**

$$\omega_{G,PSL_2(\mathbb{R})} = \omega_{WP} \tag{3.3}$$

which we rewrite in our setting:

$$(\sigma_{\mathcal{F}})^* \omega_G = \omega_{WP} \ . \tag{3.4}$$

3.5 A Lagrangian embedding

Let $\hat{M}$ be a compact 3-manifold as in section 1.5. We will use here the same notations as in section 1.5, let us briefly recall these. The boundary $\partial\hat{M}$ is the disjoint union of N surfaces S_k of genera at least 2. The Teichmüller space of the boundary is given by $\mathcal{T}(\partial\hat{M}) = \mathcal{T}(S_1) \times \cdots \times \mathcal{T}(S_N)$, and similarly $\mathcal{CP}(\partial\hat{M}) = \mathcal{CP}(S_1) \times \cdots \times \mathcal{CP}(S_N)$. The forgetful projection is the holomorphic map $p = p_1 \times \cdots \times p_N : \mathcal{CP}(\partial\hat{M}) \to \mathcal{T}(\partial\hat{M})$, and $\beta : \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$ is the "simultaneous uniformization section".

By Goldman's construction discussed above, $\mathcal{CP}(\partial\hat{M})$ is equipped with a complex symplectic structure ω_G, which is obtained here as

$$\omega_G = {\mathrm{pr}_1}^* {\omega_G}^{(1)} + \cdots + {\mathrm{pr}_N}^* {\omega_G}^{(N)} \ , \tag{3.5}$$

where ${\omega_G}^{(k)}$ is the complex symplectic structure on $\mathcal{CP}(S_k)$ and pr_k is the k^{th} projection map $\mathcal{CP}(\partial\hat{M}) \to \mathcal{CP}(S_k)$.

There is a general argument, discovered in this setting by Steven Kerckhoff, which shows that

Theorem 3.3. *$\beta : \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$ is a Lagrangian embedding.*

Although this is a consequence of our theorem 5.14, we briefly explain this nice argument, based on Poincaré duality in cohomology. This could be done directly on the manifolds $\mathcal{HC}(M)$ and $\mathcal{CP}(\partial\hat{M})$, but we prefer to transport the situation to character varieties, where it is simpler.

Recall that the simultaneous uniformization section β was defined as the composite $\beta = \psi \circ \varphi^{-1}$, where $\psi : \mathcal{HC}(M) \to \mathcal{CP}(\partial\hat{M})$ is the map which assigns the induced projective structure on $\partial\hat{M}$ to each cocompact hyperbolic structure on the interior M of $\hat{M}$, and $\varphi = p \circ \psi : \mathcal{HC}(M) \to \mathcal{T}(\partial\hat{M})$ is a biholomorphism.

By definition, the embedding β is Lagrangian if it is isotropic ($\beta^*\omega_G = 0$) and $\dim \mathcal{CP}(\partial\hat{M}) = 2\dim \mathcal{T}(\partial\hat{M})$. We already know the second statement to be true (see section 1.2). It remains to show that β is isotropic, but since ϕ is a diffeomorphism, this amounts to showing that $\psi : \mathcal{HC}(M) \to \mathcal{CP}(\partial\hat{M})$ is isotropic.

Let us have a look at the equivalent statement on holonomy: there is a commutative diagram

$$\begin{array}{ccc} \mathcal{HC}(M) & \xrightarrow{\quad\psi\quad} & \mathcal{CP}(\partial\hat{M}) \\ \downarrow{\scriptstyle hol} & & \downarrow{\scriptstyle hol} \\ \hat{\mathcal{X}}(M, PSL_2(\mathbb{C})) & \xrightarrow{\quad f\quad} & \mathcal{X}(\partial\hat{M}, PSL_2(\mathbb{C})) \end{array} ,$$

where $f : \mathcal{X}(M, PSL_2(\mathbb{C})) \to \mathcal{X}(\partial\hat{M}, PSL_2(\mathbb{C}))$ is the map between character varieties induced by the "restriction" map $\iota_* : \pi_1(\partial\hat{M}) \to \pi_1(\hat{M})$ [8]. Since the property of being isotropic is local, it is enough to show the following proposition:

Proposition 3.4. *The map $f : \mathcal{X}(\hat{M}, PSL_2(\mathbb{C})) \to \mathcal{X}(\partial\hat{M}, PSL_2(\mathbb{C}))$ is isotropic.*

Proof. Let $[\rho] \in \mathcal{X}(\hat{M}, PSL_2(\mathbb{C}))$. The map

$$df : T_{[\rho]}\mathcal{X}(\hat{M}, PSL_2(\mathbb{C}) \to T_{[\rho\circ\iota_*]}\mathcal{X}(\partial\hat{M}, PSL_2(\mathbb{C}))$$

is the map α that appears in long exact sequence in cohomology of the pair $(M, \partial M)$ as follows. This exact diagram shows a piece of this sequence written in terms of group cohomology, where vertical arrows are given by Poincaré duality:

$$\begin{array}{ccccc} H^1(\pi_1(\hat{M}), \mathfrak{g}_{\mathrm{Ad}\circ\rho}) & \xrightarrow{\alpha} & H^1(\pi_1(\partial\hat{M}), \mathfrak{g}_{\mathrm{Ad}\circ\rho}) & \xrightarrow{\beta} & H^2(\pi_1(\hat{M}), \pi_1(\partial\hat{M}); \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(\pi_1(\hat{M}), \pi_1(\partial\hat{M}); \mathfrak{g}_{\mathrm{Ad}\circ\rho})^* & \xrightarrow{\alpha^*} & H^1(\pi_1(\partial\hat{M}), \mathfrak{g}_{\mathrm{Ad}\circ\rho})^* & \xrightarrow{\beta^*} & H^1(\pi_1(\partial\hat{M}), \mathfrak{g}_{\mathrm{Ad}\circ\rho})^* \end{array}$$

Note that if $u \in H^1(\pi_1(\partial\hat{M}), \mathfrak{g}_{\mathrm{Ad}\circ\rho})$, the Poincaré dual of u is defined by the relation $\langle u^*, v\rangle = \tilde{B}(u \cup v) \cap [\partial\hat{M}]$ for all $v \in H^1(\pi_1(\partial\hat{M}), \mathfrak{g}_{\mathrm{Ad}\circ\rho})$, where $[\partial\hat{M}]$ is the fundamental class of $\partial\hat{M}$. This is precisely saying that $\langle u^*, v\rangle = \omega_G(u, v)$. It follows that α is isotropic: using the commutativity and exactness of the diagram, we can write

$$\begin{aligned} \omega_G(\alpha(u), \alpha(v)) &= \langle \alpha(u)^*, \alpha(v)\rangle \\ &= \langle \beta^*(u^*), \alpha(v)\rangle \\ &= \langle u^*, \beta \circ \alpha(v)\rangle \\ &= 0\ . \end{aligned}$$

8. Note that if $\partial\hat{M}$ is disconnected, we define its fundamental group $\pi_1(\partial\hat{M})$ as the free product of the fundamental groups of its components, so that a representation $\rho : \pi_1(\partial\hat{M}) \to PSL_2(\mathbb{C})$ is just a N-tuple of representations $\rho_k : \pi_1(S_k) \to PSL_2(\mathbb{C})$.

□

Remark 3.5. Note that in the quasifuchsian situation $M = S \times \mathbb{R}$, Theorem 3.3 is trivial, or rather its formulation in terms of holonomy (cf. Proposition 3.4 above). Indeed, the map $f : \mathcal{X}(\hat{M}, PSL_2(\mathbb{C})) \to \mathcal{X}(\partial\hat{M}, PSL_2(\mathbb{C}))$ in that case is just the diagonal embedding of $\mathcal{X}(\pi, PSL_2(\mathbb{C}))$ [9] into $\mathcal{X}(\pi, PSL_2(\mathbb{C})) \times \mathcal{X}(\pi, PSL_2(\mathbb{C}))$ [10].

9. where $\pi = \pi_1(\hat{M}) = \pi_1(S)$.

10. Here $\mathcal{X}(\pi, PSL_2(\mathbb{C})) \times \mathcal{X}(\pi, PSL_2(\mathbb{C}))$ is equipped with with the complex symplectic structure ${\mathrm{pr}_1}^*\omega_G - {\mathrm{pr}_2}^*\omega_G$ (the minus sign is due to the opposite orientation of $\partial^+\hat{M}$ and $\partial^-\hat{M}$). The fact that the diagonal is Lagrangian is a particular case of the following general fact: if (X, ω) is a symplectic manifold and $X \times X$ is equipped with the symplectic structure ${\mathrm{pr}_1}^*\omega - {\mathrm{pr}_2}^*\omega$, then the graph of a function $h : X \to X$ is a Lagrangian submanifold of $X \times X$ if and only if h is a symplectomorphim.

CHAPTER 4

Complex Fenchel-Nielsen coordinates and Platis' symplectic structure

4.1 Fenchel-Nielsen coordinates on Teichmüller space and Wolpert theory

Pants decomposition and Fenchel-Nielsen coordinates

Let S be a closed connected oriented surface of genus $g \geqslant 2$. In this section, we are not going to talk about the Teichmüller space of S as we have defined it in section 1.1, but rather the Fricke space $\mathcal{F}(S)$. Recall that $\mathcal{F}(S)$ is the set of marked hyperbolic structures on S (or marked Fuchsian projective structures), which is of course analytically diffeomorphic to $\mathcal{T}(S)$ as we have seen in section 3.4.

Let us first briefly recall the construction of the classical Fenchel-Nielsen coordinates on $\mathcal{F}(S)$. These depend on the choice of a *pants decomposition* of S, *i.e.* an ordered maximal collection of distinct, disjoint [1], nontrivial free homotopy classes of simple [2] closed curves $\alpha = (\alpha_1, \dots, \alpha_N)$.

The following are classical facts:

- $N = 3g - 3$.
- If $c_1, \dots, c_N$ are disjoint representatives of $\alpha_1, \dots, \alpha_N$ (respectively), then $S \setminus \bigcup_{i=1}^{N} c_i$ is a disjoint union of $M = 2g - 2$ topological pair of pants P_k (thrice-punctured spheres).
- If X is a hyperbolic structure on S, every nontrivial free homotopy class of

1. in the sense that for $j \neq k$, there exists representatives of α_j and α_k.
2. meaning that there exists a simple representative.

simple closed curves γ is uniquely represented by a simple a closed geodesic γ^X.

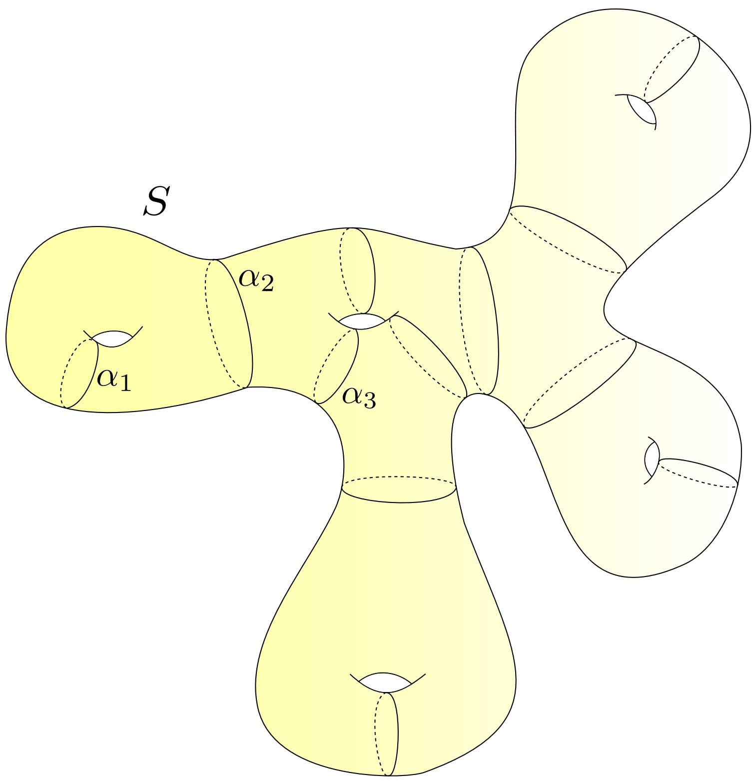

Fig.1- A pants decomposition of a surface. One can count $g = 5$, $N = 12$ and $M = 8$.

Given a hyperbolic structure X on S, denote by $l_\gamma(X)$ the hyperbolic length of γ^X. This defines a *length function* $l_\gamma : \mathcal{F}(S) \to \mathbb{R}_{>0}$. In particular, given a pants decomposition α, one gets a function

$$l_\alpha : \mathcal{F}(S) \to (\mathbb{R}_{>0})^N \ . \tag{4.1}$$

The components l_{α_i} of l_α are called the *Fenchel-Nielsen length parameters.*

Any hyperbolic structure X on S determines a hyperbolic structure (with geodesic boundary) on each one of the closed pair of pants $\overline{P_k}$ in the decomposition $S \setminus \bigcup_{i=1}^N \alpha_i^X = \bigsqcup_{k=1}^M P_k$. It is well-known that a hyperbolic structure on a closed pair of pants is uniquely determined by the lengths of its three boundary components. This follows from the observation that a hyperbolic pair of pants is obtained by gluing two isometric oppositely oriented right-angled hexagons in $\mathbb{H}^2$ and the following elementary theorem in plane hyperbolic geometry:

Proposition 4.1. *Up to isometry, there exists a unique right-angled hexagon in $\mathbb{H}^2$ with prescribed lengths on every other side.*

As a consequence, a hyperbolic structure on S is completely determined by the lengths of the curves α_i, and the parameters τ_i that prescribe how the gluing occurs along these curves, *i.e.* by which amount of "twisting". However, these parameters τ_i are not very well defined: there is no obvious choice of the hyperbolic structure obtained by "not twisting at all before gluing". Also, note that assuming that such a choice is made, each of these parameters should live in $\mathbb{R}$ indeed and not $\mathbb{R}/2\pi\mathbb{Z}$: although there is a natural isometry $f : X \to Y$ where Y is obtained by 2π-twisting X along some curve α_i, f is not homotopic to the identity.

Let us make this more precise. For any nontrivial free homotopy class of simple closed curves γ, there is a flow (an $\mathbb{R}$-action) called *twisting along* γ

$$\mathrm{tw}_\gamma : \mathbb{R} \times \mathcal{F}(S) \to \mathcal{F}(S) \ . \tag{4.2}$$

The flow is freely transitive in the fibers of l_γ. Let us mention that twist deformations along simple closed curves are naturally generalized first to weighted multicurves, then to the completion $\mathcal{ML}(S)$ of measured laminations. This generalization is the notion of *earthquake* introduced by Thurston (see e.g. [Ker83]).

Denote by tw_α the $\mathbb{R}^N$-action $\mathrm{tw}_\alpha = (\mathrm{tw}_{\alpha_1}, \dots, \mathrm{tw}_{\alpha_N}) : \mathbb{R}^N \times \mathcal{F}(S) \to \mathcal{F}(S)$. The fact that a hyperbolic structure on S is uniquely determined by the lengths parameters l_{α_i} and the amount of twisting a long each α_i is precisely stated as: the $\mathbb{R}^N$-action tw_α is freely transitive in the fibers of l_α, and the reunion of these fibers is the whole Fricke space $\mathcal{F}(S)$. In particular,

Theorem 4.2. *Choosing a smooth section to l_α determines a diffeomorphism $(l_\alpha, \tau_\alpha) : \mathcal{F}(S) \to (\mathbb{R}_{>0})^N \times \mathbb{R}^N$.*

The function τ above is naturally defined by $\mathrm{tw}_\alpha(\tau_\alpha, \sigma \circ l_\alpha) = \mathrm{id}_{\mathcal{F}(S)}$, where σ is the chosen section. The components $\tau_{\alpha_1}, \dots, \tau_{\alpha_N}$ of τ are called the *Fenchel-Nielsen twist parameters*. The theorem above thus says that Fenchel-Nielsen length and twist parameters are global coordinates on $\mathcal{F}(S)$. In particular, one recovers $\dim_{\mathbb{R}} \mathcal{T}(S) = \dim_{\mathbb{R}} \mathcal{F}(S) = 2N = 6g - 6$. It also appears that $\mathcal{T}(S) \approx \mathcal{F}(S)$ is topologically a cell, and it follows that $\mathcal{CP}(S)$ is also a cell.

Note that although the coordinates τ_{α_i} depend on the choice on a section to l_α, the 1-forms $d\tau_{\alpha_i}$ and the vector fields $\frac{\partial}{\partial \tau_{\alpha_i}}$ do not. In fact, $\frac{\partial}{\partial \tau_\gamma}$ is well-defined for any nontrivial free homotopy class of simple closed curve γ, and its flow is of course the twist flow $\mathrm{tw}_\gamma : \mathbb{R} \times \mathcal{F}(S) \to \mathcal{F}(S)$.

Wolpert theory

Let us first briefly recall a few notions of symplectic geometry and the language of Hamiltonian mechanics. If (M^{2N}, ω) is a symplectic manifold, ω determines an bundle map $\omega^\flat : TM \to T^*M$ defined by $\omega^\flat(u) = \omega(u, \cdot)$. Since ω is nondegenerate, $\omega^\flat$ is an isomorphism, its inverse is denoted by $\omega^\sharp : T^*M \to TM$. If α is a one-form on M, $\omega^\sharp(\alpha)$ is thus the unique vector field X such that $i_X\omega = \alpha$. If f is a function on M, the vector field $X_f := \omega^\sharp(df)$ is called the *Hamiltonian* (or symplectic gradient) of f. Note that a vector field X is Hamiltonian is and only if the 1-form $i_X\omega$ [3] is exact, it follows that X satisfies $\mathcal{L}_X\omega = 0$ [4] by Cartan's magic formula. Vector fields X such that $\mathcal{L}_X\omega = 0$ are the vector fields whose flows preserve ω, they are called symplectic vector fields. The *Poisson bracket* of two functions f and g is defined by $\{f, g\} = \omega(X_f, X_g)$. f and g are said to Poisson-commute (or to be in involution) if $\{f, g\} = 0$. It is easy to see that f and g Poisson-commute if and only if f is constant along the integral curves of X_g (and vice-versa). If $f = (f_1, \ldots, f_N) : M \to \mathbb{R}^N$ is a regular map such that the f_i Poisson-commute, then f is a Lagrangian fibration. Moreover, the flows of the $-X_{f_i}$ (if they are complete) define a transitive $\mathbb{R}^N$-action that is transverse to the fibers of f (the reason for the choice of this minus sign will be apparent shortly). Notice already the analogy with the lengths functions and twist flows above. Such functions f_i are said to define a *(completely) integrable Hamiltonian system* on (M, ω). As in theorem 4.2, choosing a section to f yields coordinates $g = (g_1, \ldots, g_N) : M \to \mathbb{R}^N$ [5] such that the $\mathbb{R}^N$-action is given by the flows of the $\frac{\partial}{\partial g_i}$, in other words $\frac{\partial}{\partial g_i} = -X_{f_i}$. In general though, (f_i, g_i) is not a system of Darboux coordinates [6] for ω, but the classical Arnold-Liouville theorem states that such a choice of coordinates is possible in a way that is compatible with the Lagrangian fibration and the $\mathbb{R}^N$-action (see e.g. [Dui80] for a precise statement and proof of this theorem). The Darboux coordinates obtained by Arnold-Liouville's theorem are called *action-angle coordinates.*

In [Wol82], [Wol83] and [Wol85], Wolpert developed a very nice theory describing the symplectic geometry of $\mathcal{F}(S)$ in relation to Fenchel-Nielsen coordinates. Let us present some of his results. In the following, $\mathcal{F}(S)$ is equipped with its standard symplectic structure ω_G ($= \omega_{WP}$ under the identification $\mathcal{T}(S) \approx \mathcal{F}(S)$, see section 3.4).

Theorem 4.3 (Wolpert)**.** *Let γ be any nontrivial free homotopy class of simple*

3. where $i_X\omega$ is the contraction of ω with the vector field X.
4. where $\mathcal{L}_X$ is the Lie derivative along the vector field X.
5. To be accurate, g takes values in $\mathbb{R}^{N-k} \times \mathbb{T}^k$ in general, where k is some integer and $\mathbb{T}^k$ is the k-dimensional torus.
6. By definition, (f_i, g_i) are called *Darboux coordinates* on (M, ω) if they are canonical for the symplectic structure: $\omega = \sum_{i=1}^N df_i \wedge dg_i$. The celebrated theorem of Darboux says that there always exists Darboux coordinates locally on any symplectic manifold.

closed curves on S. The flow of the Hamiltonian vector field $-X_{l_\gamma}$ is precisely the twist flow tw_γ*.*

In other words,

$$\frac{\partial}{\partial \tau_\gamma} = -X_{l_\gamma} \ . \tag{4.3}$$

Theorem 4.4 (Wolpert)**.** *Let γ and γ' be distinct nontrivial free homotopy classes of simple closed curves on S. Then at any point $X \in \mathcal{F}(S)$,*

$$\omega_G \left(\frac{\partial}{\partial \tau_\gamma}, \frac{\partial}{\partial \tau_{\gamma'}} \right) = \sum_{p \in (\gamma^X \cap \gamma'^X)} \cos \theta_p \ , \tag{4.4}$$

where θ_p is the angle between the geodesics γ^X and γ'^X at p.

A direct consequence of these two theorems is:

Theorem 4.5. *If α is a pants decomposition of S, then Fenchel-Nielsen length functions l_{α_i} define an integrable Hamiltonian system. The Hamiltonian $\mathbb{R}^N$-action associated to this system is the twist flow* tw_γ*.*

Wolpert also shows that

Proposition 4.6 (Wolpert)**.** *If α is a pants decomposition of S, then for any $i, j \in \{1, \ldots, N\}$*

$$\omega_G \left(\frac{\partial}{\partial l_{\alpha_i}}, \frac{\partial}{\partial l_{\alpha_j}} \right) = 0 \ . \tag{4.5}$$

It follows that we are in the best possible situation:

Theorem 4.7 (Wolpert)**.** *Let α be a pants decomposition of S. Fenchel-Nielsen length and twist parameters associated to α are respectively action and angle variables for the integrable Hamiltonian system defined by the functions l_{α_i}. In particular, Fenchel-Nielsen coordinates are Darboux coordinates for the symplectic structure:*

$$\omega_G = \sum_{i=1}^{N} dl_{\alpha_i} \wedge d\tau_{\alpha_i} \ . \tag{4.6}$$

It is remarkable in particular that this does not depend on the choice of the pants decomposition α.

4.2 Complex Fenchel-Nielsen coordinates

Kourouniotis (in [Kou94], see also [Kou91] and [Kou92]) and Tan (in [Tan94]) introduced a system of global holomorphic coordinates $(l^{\mathbb{C}}, \tau^{\mathbb{C}}) : \mathcal{QF}(S) \to \mathbb{C}^N \times \mathbb{C}^N$ that can be thought of as a complexification of Fenchel-Nielsen coordinates on the Fuchsian slice $\mathcal{F}(S)$. Let us outline this construction. We refer to [Kou94], [Tan94] and also [Ser01] for details.

Complex distance and displacement in hyperbolic space

Let α and β be two geodesics in the hyperbolic space $\mathbb{H}^3$. The *complex distance* between α and β is the complex number $\sigma = \sigma(\alpha, \beta)$ (defined modulo $2i\pi\mathbb{Z}$) such that $\mathrm{Re}(\sigma)$ is the hyperbolic distance between α and β and $\mathrm{Im}(\sigma)$ is the angle between them (meaning the angle between the two planes containing their common perpendicular and either α or β). In the upper half-space model $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+^*$, after applying an isometry so that α has endpoints $(u, -u)$ and β has endpoints $(p, -p)$ (where $u, p \in \mathbb{C}\mathbf{P}^1$), σ is determined by $e^{\sigma} u = p$. Note that one has to be careful about orientations and sign to define σ unambiguously, see [Kou94] and [Ser01] for details.

Let f be a non-parabolic isometry of $\mathbb{H}^3$ different from the identity, and β a geodesic perpendicular to the axis of f. The *complex displacement* of f is the complex distance φ between β and $f(\beta)$. If f is represented by a matrix $A \in SL_2(\mathbb{C})$, the complex displacement of f is given by

$$2\cosh\left(\frac{\varphi}{2}\right) = \mathrm{tr}(A) \; . \tag{4.7}$$

The complex displacement and oriented axis of a non-parabolic isometry determine it uniquely.

Right-angled hexagons and pair of pants in hyperbolic space

An (oriented skew) *right-angled* hexagon in $\mathbb{H}^3$ is a cyclically ordered set of six oriented geodesics α_k indexed by $k \in \mathbb{Z}/6\mathbb{Z}$, such that α_k intersects α_{k+1} orthogonally. Define the complex length of the "side" α_k by $\sigma_k = \sigma(\alpha_{k-1}, \alpha_{k+1})$.

Proposition 4.8. *The following relations are showed in [Fen89]:*
Sine rule:

$$\frac{\sinh\sigma_1}{\sinh\sigma_4} = \frac{\sinh\sigma_3}{\sinh\sigma_6} = \frac{\sinh\sigma_5}{\sinh\sigma_2} \tag{4.8}$$

Cosine rule:

$$\cosh\sigma_n = \frac{\cosh\sigma_{n+3} - \cosh\sigma_{n+1}\cosh\sigma_{n-1}}{\sinh\sigma_{n+1}\sinh\sigma_{n-1}} \tag{4.9}$$

Using these formulas, one shows that assigning complex lengths on every other side determines a unique right-angled hexagon in $\mathbb{H}^3$ up to (possibly orientation-reversing) isometry. In [Kou94] and [Tan94], it is showed the the construction of a hyperbolic pair of pants by gluing two right-angled hexagons can be extended to $\mathbb{H}^3$. Such a pair of pants is thus uniquely determined by the complex lengths of its boundary components. In terms of holonomy ([Kou94]):

Proposition 4.9. *Let P be a topological pair of pants and σ_1, σ_2, $\sigma_3 \in \mathbb{C}_+$ (i.e. with $\mathrm{Re}(\sigma_i) > 0$). There is a unique representation up to conjugation*

$$\rho : \pi_1(P) = \langle c_1, c_2, c_3 \mid c_1 c_2 c_3 = 1 \rangle \to PSL_2(\mathbb{C})$$

such that $\mathrm{tr}(\rho(c_i)) = -2\cosh \sigma_i$.

Complex lengths and complex twisting in the quasifuchsian space

Let $Z \in \mathcal{QF}(S)$ be a quasifuchsian structure on S and $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ its holonomy representation. For any nontrivial free homotopy class of simple closed curves γ, define the *complex length* of γ as the complex displacement of the hyperbolic isometry $\rho(\gamma)$. This defines a holomorphic function $l^{\mathbb{C}}_\gamma : \mathcal{QF}(S) \to \mathbb{C}_+$. In the quasifuchsian 3-manifold M, $\rho(\gamma)$ corresponds to a geodesic of complex length $l^{\mathbb{C}}_\gamma$, *i.e.* of hyperbolic length $\mathrm{Re}(l^{\mathbb{C}}_\gamma)$ and torsion $\mathrm{Im}(l^{\mathbb{C}}_\gamma)$. It is easy to see that if Z is a Fuchsian structure, then $l^{\mathbb{C}}_\gamma(Z) = l_\gamma(Z)$. If $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a pants decomposition of S, we call

$$l^{\mathbb{C}}_\alpha = (l^{\mathbb{C}}_{\alpha_1}, \ldots, l^{\mathbb{C}}_{\alpha_N}) : \mathcal{QF}(S) \to (\mathbb{C}_+)^N \tag{4.10}$$

the *complex Fenchel-Nielsen length parameters.*

As a consequence of the previous discussion, if the complex lengths $l^{\mathbb{C}}_{\alpha_1}, \ldots, l^{\mathbb{C}}_{\alpha_N}$ are fixed, a quasifuchsian structure on S is determined by how the gluings of pair of pants occur along their common boundaries. Analogously to the Fuchsian case, this is prescribed by a complex parameter $\tau^{\mathbb{C}}_{\alpha_i}$, that we will call a complex twist parameter , that describes both the amount of twisting (by $\mathrm{Re}(\tau^{\mathbb{C}}_{\alpha_i})$) and the amount of bending (by $\mathrm{Im}(\tau^{\mathbb{C}}_{\alpha_i})$) before gluing. $\tau^{\mathbb{C}}_{\alpha_i}$ can be more or less well defined as the complex distance between two adequate geodesics in $\mathbb{H}^3$, but the definition is clearer in terms of the effect of complex-twisting by $\tau^{\mathbb{C}}_{\alpha_i}$ on the holonomy of the glued pairs of pants (see [Kou94], [Gol04]).

As in the Fuchsian case, it is the complex twist flow $\mathrm{tw}^{\mathbb{C}}_\gamma$ along a simple closed curve γ that is well-defined rather than the twist parameter $\tau^{\mathbb{C}}_{\alpha_i}$ itself, although the complex twist vector field $\dfrac{\partial}{\partial \tau^{\mathbb{C}}_\gamma}$ is well-defined. Let us mention this flow is called *bending* by Kourouniotis and corresponds to (or is a generalization of) what other authors have called *quakebends* or *complex earthquakes* discussed by Epstein-Marden

[EM87], Goldman [Gol04], McMullen [McM98], Series [Ser01] among others. It is not hard to see that starting from a Fuchsian structure Z, complex twisting by $t = t_1 + it_2 \in \mathbb{C}$ is described as the composition of twisting by t_1 on $\mathcal{F}(S)$ and then *projective grafting* by t_2 (see e.g. [Dum09] for a presentation of projective grafting).

Choosing a holomorphic section to $l_\alpha^{\mathbb{C}}$ determines *complex twist coordinates* $\tau_\alpha^{\mathbb{C}} = (\tau_{\alpha_1}^{\mathbb{C}}, \ldots, \tau_{\alpha_N}^{\mathbb{C}}) : \mathcal{QF}(S) \to \mathbb{C}^N$. We will call $(l_\alpha^{\mathbb{C}}, \tau_\alpha^{\mathbb{C}})$ complex Fenchel-Nielsen coordinates. The conclusion of our discussion is the theorem:

Theorem 4.10 (Kourouniotis, Tan)**.** *The complex Fenchel-Nielsen coordinates* $(l_\alpha^{\mathbb{C}}, \tau_\alpha^{\mathbb{C}})$ *are global holomorphic coordinates on* $\mathcal{QF}(S)$*. They restrict to the classical Fenchel-Nielsen coordinates* (l_α, τ_α) *on the Fuchsian slice* $\mathcal{F}(S)$*.*

4.3 Platis' symplectic structure

In [Pla01], Platis develops a complex version of Wolpert's theory on the quasi-fuchsian space. We recall some of his results.

First there is a complex version of Wolpert's formula 4.4:

Theorem 4.11. *There exists a complex symplectic structure* ω_P *on* $\mathcal{QF}(S)$ *such that if* γ *and* γ' *are distinct nontrivial free homotopy classes of simple closed curves on* S*, then at any point* $Z \in \mathcal{QF}(S)$ *with holonomy* ρ

$$\omega_P\left(\frac{\partial}{\partial \tau_\gamma^{\mathbb{C}}}, \frac{\partial}{\partial \tau_{\gamma'}^{\mathbb{C}}}\right) = \sum_{p \in (\gamma \cap \gamma')} \cosh \sigma_p \ , \tag{4.11}$$

where σ_p *is the complex distance between the geodesics* $\rho(\gamma)$ *and* $\rho(\gamma')$*.*

He also shows the complex analogous of theorem 4.3 in the complex symplectic manifold $(\mathcal{QF}(S), \omega_P)$:

Theorem 4.12. *Let* γ *be any nontrivial free homotopy class of simple closed curves on* S*. The complex flow of the Hamiltonian vector field* $-X_{l_\gamma^{\mathbb{C}}}$ *is precisely the complex twist flow* $\mathrm{tw}_\gamma^{\mathbb{C}}$*.*

As in the Fuchsian case, it follows from these two theorems that complex Fenchel-Nielsen length functions associated to a pants decomposition define a complex Hamiltonian integrable system. Furthermore, he proves that the striking theorem 4.7 is still true in its complex version on $(\mathcal{QF}(S), \omega_P)$:

Theorem 4.13. *If* α *is any pants decomposition of* S*, complex Fenchel-Nielsen coordinates are Darboux coordinates for the complex symplectic structure* ω_P*:*

$$\omega_P = \sum_{i=1}^{N} dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}} \ . \tag{4.12}$$

CHAPTER 5

Comparing symplectic structures

5.1 Analytic continuation

We are going to show the following proposition, which implies that two complex symplectic structures agree on $\mathcal{CP}(S)$ if and only if they agree in restriction to tangent vectors to the Fuchsian slice $\mathcal{F}(S)$:

Proposition 5.1. *Let ω be a closed $(2,0)$-form on $\mathcal{CP}(S)$ and $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S)$ be the Fuchsian section (as in (1.4)). If ${\sigma_{\mathcal{F}}}^*\omega$ vanishes on $\mathcal{T}(S)$, then ω vanishes on $\mathcal{CP}(S)$.*

The proof of this proposition is based on analytic continuation. In order to use this argument, we recall a few definitions and show some elementary facts regarding totally real submanifolds of complex manifolds.

Definition 5.2. Let M be a complex manifold and $N \subset M$ be a real submanifold. N is called *totally real* if the following holds:

$$\forall x \in N, \quad T_x N \cap J T_x N = \{0\} \ , \tag{5.1}$$

where J is the almost complex structure on M.

If moreover, N has maximal dimension $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} M$, we say that N is a *maximal totally real submanifold* of M. There are several characterizations of maximal totally real analytic submanifolds, seemingly stronger than the definition, as in the following:

Proposition 5.3. *Let M be a complex manifold of dimension n and $N \subset M$ be a real submanifold. The following are equivalent:*

– *(i) N is a maximal totally real analytic submanifold of M.*
– *(ii) $N \subset M$ locally looks like $\mathbb{R}^n \subset \mathbb{C}^n$. More precisely: for any $x \in N$, there is a holomorhic chart $z : U \to V$ where U is an open set in M containing x and V is an open set in $\mathbb{C}^n$, such that $z(U \cap N) = V \cap \mathbb{R}^n$.*
– *(iii) There is an antiholomorphic involution $\chi : M' \to M'$ where M' is a neighborhood of N in M, such that N is the set of fixed points of χ.*

If N satisfies one (equivalently all) of these conditions, M is said to be a *complexification* of N. Let us mention that any real-analytic manifold can be complefified.

Proof. It is fairly easy to see that both (ii) and (iii) imply (i), and that in fact (ii) and (iii) are equivalent. Let us show that (i) implies (ii). Using holomorphic charts, it is clearly enough to prove this in the case where N is a maximal totally real analytic submanifold of $\mathbb{C}^n$. Let $m \in N \subset \mathbb{C}^n$, there is a real-analytic parametrization $\varphi : D \to N$, where D is a small open disk centered at the origin in $\mathbb{R}^n$, such that $\varphi(0) = m$ and $D\varphi(0) \neq 0$. The map φ is given by a convergent power series $\varphi(x) = \sum_{|\alpha|=n} a_\alpha x^\alpha$ for all $x \in D$, where the sum is taken over all multi-indices α of length n, and the a_α are coefficients in $\mathbb{C}^n$. In order to extend φ to a holomorphic map $\Phi : D' \to M$ where D' is the disk in $\mathbb{C}^n$ such that $D = D' \cap \mathbb{R}^n$, we can just replace $x \in D$ by $z \in D'$ in the expression of φ: define $\Phi(z) = \sum_{|\alpha|=n} a_\alpha z^\alpha$. This power series converges in D' because it has the same radius of convergence as its real counterpart. Moreover, if D' is small enough, Φ is a biholomorphism onto its image because $d\Phi(0) = d\varphi(0) \neq 0$. This shows that (i) implies (ii) (just take the chart given by Φ^{-1}). Note that the simplicity of this proofs hides a little trick: the actual complexification of φ is a map $\varphi : D' \to \mathbb{C}^{2n}$ (and not $\mathbb{C}^n$), when a_α is seen as a real vector in $\mathbb{C}^n$. □

Keeping in mind that we want to consider the Fuchsian slice in $\mathcal{CP}(S)$, we make this last general observation on totally real submanifolds:

Proposition 5.4. *Let V be a complex manifold. The diagonal Δ in $V \times \overline{V}$*[1] *is a maximal totally real analytic submanifold.*

Proof. This is a direct consequence of characterization (iii) in the previous proposition: just take the antiholomorphic involution $\chi : V \times \bar{V} \to V \times \bar{V}$ defined by $\chi(x, y) = (y, x)$. □

An immediate application of this is that the Fuchsian slice $\mathcal{F}(S)$ is a maximal totally real analytic submanifold of $\mathcal{CP}(S)$ (as was already pointed out in section

1. $\overline{V}$ denotes the manifold V equipped with the opposite complex structure.

3.4): it is the image of the diagonal of $\mathcal{T}(S) \times \mathcal{T}(\overline{S})$ [2] by the holomorphic embedding β^+ (see section 1.4). Another way to see this is that the quasifuchsian space $\mathcal{QF}(S) = \mathrm{Im}(\beta^+) \subset \mathcal{CP}(S)$ is equipped with a canonical antiholomorphic involution, which justs consists in "turning a quasifuchsian 3-manifold upside down", and $\mathcal{F}(S)$ is the set of fixed points of this involution.

Now, we prove this first elementary analytic continuation theorem:

Proposition 5.5. *Let M be a connected complex manifold and $N \subset M$ be a maximal totally real submanifold. If $f : M \to \mathbb{C}$ is a holomorphic function that vanishes on N, then f vanishes on M.*

Proof. By the identity theorem for holomorphic functions, it is enough to show that f vanishes on a small open neighborhood U of some point $x \in N$. If N is analytic, this is an straightforward consequence of characterization (ii) in Proposition 5.3. Let us produce a proof that does not assume analyticity of N. Since the restriction $f_{|N}$ vanishes identically, we have $(df)_{|TN} = 0$. Using the fact that $T_xM = T_xN \oplus JT_xN$ for all $x \in N$ and the holomorphicity of f, it is easy to derive that df vanishes at all points of N. In particular, if $z = (z_k)_{1 \leqslant k \leqslant n} : U \to \mathbb{C}^n$ is a holomorphic chart, the partial derivatives $\dfrac{\partial f}{\partial z_k}$ vanish on N. But those are again holomorphic functions, so we can use the same argument: their partial derivatives must vanish on N. By an obvious induction, we see that all partial derivatives of f (at any order) vanish at points of N. Since f is holomorphic, this implies that $f = 0$. $\square$

We can now finally prove:

Proposition 5.6. *Let M be a connected complex manifold and $\sigma : N \to M$ be a maximal totally real embedding. If ω is a closed $(2,0)$-form on M such that $\sigma^*\omega = 0$, then $\omega = 0$.*

Proof. We can suppose that $N \subset M$, the hypothesis is that $\omega_{|TN} = 0$. Since $T_xM = T_xN \oplus JT_xN$ for any $x \in N$ and ω is of type $(2,0)$, it is easy to see that ω vanishes at points of N. Now, recall that a closed $(2,0)$-form is holomorphic. Let $z = (z_k)_{1 \leqslant k \leqslant n} : U \to \mathbb{C}^n$ be a holomorphic chart in a neighborhood of a point $x \in N$, ω has an expression of the form $\omega = \sum_{j,k} f_{jk} dz_j \wedge dz_k$ where f_{jk} are holomorphic functions on U. Since ω vanishes at points of N, the functions f_{jk} vanish on $U \cap N$, and we derive from the previous proposition that they actually vanish on U. We thus have $\omega_{|U} = 0$, and it follows once again from the identity theorem (taken in charts) that ω vanishes on M. $\square$

2. Recall that $\mathcal{T}(\overline{S})$ is canonically identified with $\overline{\mathcal{T}(S)}$.

An immediate consequence of this, together with (3.4), is that a complex symplectic structure on $\mathcal{CP}(S)$ agrees with the standard complex symplectic structure if and only if it induces the Weil-Petersson Kähler form on the Fuchsian slice:

Theorem 5.7. *Let ω be a complex symplectic structure on $\mathcal{CP}(S)$. Then $\omega = \omega_G$ if and only if $(\sigma_{\mathcal{F}})^* \omega = \omega_{WP}$.*

Proof. By the previous proposition, $\omega = \omega_G$ if and only if $(\sigma_{\mathcal{F}})^*(\omega - \omega_G) = 0$. But by (3.4), $\sigma_{\mathcal{F}}^* \omega_G = \omega_{WP}$, therefore $(\sigma_{\mathcal{F}})^*(\omega - \omega_G) = (\sigma_{\mathcal{F}})^*\omega - \omega_{WP}$. □

5.2 The affine cotangent symplectic structures

Recall (see section 2.3) that any section $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ determines an affine identification $\tau^\sigma : \mathcal{CP}(S) \xrightarrow{\sim} T^*\mathcal{T}(S)$ and thus a complex-valued nondegenerate 2-form $\omega^\sigma = (\tau^\sigma)^*\omega$ on $\mathcal{CP}(S)$. ω^σ is a complex symplectic structure on $\mathcal{CP}(S)$ if and only if σ is a holomorphic section to p. We will now answer the question: for which holomorphic sections σ does ω^σ agree with the standard symplectic structure ω_G?

As a direct consequence of theorem 5.7, together with Proposition 2.3, we show:

Theorem 5.8. *Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be a section to p. Then ω^σ agrees with the standard complex symplectic structure ω_G on $\mathcal{CP}(S)$ if and only if $\sigma_{\mathcal{F}} - \sigma$ is a primitive for the Weil-Petersson metric on $\mathcal{T}(S)$:*

$$\omega^\sigma = \omega_G \;\Leftrightarrow\; d(\sigma_{\mathcal{F}} - \sigma) = \omega_{WP}\ . \tag{5.2}$$

More generally, if c is some complex constant,

$$\omega^\sigma = c\omega_G \;\Leftrightarrow\; d(\sigma_{\mathcal{F}} - \sigma) = c\omega_{WP}\ . \tag{5.3}$$

Proof. By Theorem 5.7, $\omega^\sigma = \omega_G$ if and only if $(\sigma_{\mathcal{F}})^*\omega^\sigma = \omega_{WP}$. However, it follows from Proposition 2.3 that

$$\begin{aligned}
(\sigma_{\mathcal{F}})^*\omega^\sigma &= (\sigma_{\mathcal{F}})^* \left[\omega^\sigma_{\mathcal{F}} - p^*d(\sigma - \sigma_{\mathcal{F}})\right] \\
&= (\sigma_{\mathcal{F}})^*((\tau^{\sigma_{\mathcal{F}}})^*\omega) - (\sigma_{\mathcal{F}})^*(p^*d(\sigma - \sigma_{\mathcal{F}})) \\
&= (\tau^{\sigma_{\mathcal{F}}} \circ \sigma_{\mathcal{F}})^*\omega - (p \circ \sigma)^* d(\sigma - \sigma_{\mathcal{F}}) \\
&= {s_0}^*\omega - \mathrm{id}^* d(\sigma - \sigma_{\mathcal{F}}) \\
&= -d(\sigma - \sigma_{\mathcal{F}})\ .
\end{aligned}$$

Let us make a couple of comments on this calculation: recall that ω denotes the canonical symplectic structure on $T^*\mathcal{T}(S)$; $\tau^\sigma \circ \sigma = s_0$ is the characterization of τ^σ; and ${s_0}^*\omega = 0$ because the zero section s_0 is a Lagrangian in $T^*\mathcal{T}(S)$ (see section 2.2). Of course, the proof of the apparently more general second statement is just the same. □

Now, McMullen proved in [McM00] the following theorem:

Theorem 5.9 (McMullen [McM00])**.** *If σ is any Bers section, then*

$$d(\sigma_{\mathcal{F}} - \sigma) = -i\omega_{WP} \ . \tag{5.4}$$

Using this result, we eventually obtain as a corollary of Theorem 5.8:

Theorem 5.10. *If $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ is any Bers section, then*

$$\omega^{\sigma} = -i\omega_G \ . \tag{5.5}$$

In particular, we can deduce from this identification the following properties:

Corollary 5.11. *Consider the space $\mathcal{CP}(S)$ equipped with its standard symplectic structure ω_G. Then*

1. *The canonical projection $p : \mathcal{CP}(S) \to \mathcal{T}(S)$ is a Lagrangian fibration.*
2. *Bers slices are the leaves of a Lagrangian foliation of the quasifuchsian space $\mathcal{QF}(S)$.*

We also derive an explicit expression of $\omega_G(u, v)$ when u is a vertical tangent vector (by Proposition 2.4):

Corollary 5.12. *Let u, v be tangent vectors at $Z \in \mathcal{CP}(S)$ such that u is vertical, i.e. $p_* u = 0$. Then*

$$\omega_G(u, v) = i\langle u, p_* v\rangle \ . \tag{5.6}$$

Looking back at Proposition 2.3, we also get the expression of the 2-form $\omega_{\mathcal{F}}$ obtained under the Fuchsian identification:

Corollary 5.13. *Let $\sigma_{\mathcal{F}} : \mathcal{T}(S) \to \mathcal{CP}(S)$ be the Fuchsian section. Then*

$$\omega^{\sigma_{\mathcal{F}}} = -i(\omega_G - p^*\omega_{WP}) \ . \tag{5.7}$$

It should not come as a surprise that we see from this equality that $\omega^{\sigma_{\mathcal{F}}}$ vanishes on the Fuchsian slice. Notice the equality between real symplectic structures:

$$\mathrm{Re}(\omega^{\sigma_{\mathcal{F}}}) = \mathrm{Im}(\omega_G) \tag{5.8}$$

and that $\mathrm{Re}(\omega^{\sigma_{\mathcal{F}}})$ is (half) the real canonical symplectic structure on $T^*\mathcal{T}(S)$ pulled back by the Fuchsian identification.

Finally, let us mention that McMullen's *quasifuchsian reciprocity* theorem showed in [McM00] can be seen as a consequence of Theorem 5.10. We will give a precise statement and proof of a generalized version of this theorem in the setting of convex cocompact 3-manifolds (Theorem 5.18).

Generalizations in the setting of convex cocompact hyperbolic 3-manifolds

Let $\hat{M}$ be a compact 3-manifold as in section 1.5. We will use here the same notations as in 1.5. Recall that we have defined there a canonical holomorphic section $\beta : \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$.

McMullen and Takhtajan-Teo gave generalized versions of quasifuchsian reciprocity, which they called *Kleinian reciprocity*, notably in [McM00] (Appendix) and [TT03]. In particular,**Theorem 6.3** in [TT03] says the following:

Theorem. Let $\sigma_{\mathcal{F}} : \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$ denote the Fuchsian section. Then

$$d(\sigma_{\mathcal{F}} - \beta) = -i\omega_G \ .$$

Since our theorem 5.8 above does not assume that S is connected, we obtain:

Theorem 5.14. *Let ω_G be the standard complex symplectic structure on $\mathcal{CP}(\partial\hat{M})$ and $\omega^\beta = (\tau^\beta)^*\omega$ be the complex symplectic structure obtained by the identification $\tau^\beta : \mathcal{CP}(\partial\hat{M}) \xrightarrow{\sim} T^*\mathcal{T}(\partial\hat{M})$ as in section 2.3. Then*

$$\omega^\beta = -i\omega_G \ . \tag{5.9}$$

A first immediate corollary is that we recover Theorem 3.3: β is a Lagrangian embedding.

Another consequence of this theorem and of the fact that the projections $p_k : \mathcal{CP}(S_k) \to \mathcal{T}(S_k)$ are Lagrangian (Theorem 5.11) is a generalization of Theorem 5.10:

Theorem 5.15. *Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section (see section 1.5). Then*

$$\omega^\sigma = -i\omega_G \ . \tag{5.10}$$

Proof. By definition, σ is map defined by $\sigma = f_{(X_i),j}$ as in section 1.5, where $S = S_j$ and X_i is a fixed point in $\mathcal{T}(S_i)$ for $i \neq j$. Recall that

$$\omega_G = (\mathrm{pr}_1)^*{\omega_G}^{(1)} + \cdots + (\mathrm{pr}_N)^*{\omega_G}^{(N)}$$

where ${\omega_G}^{(k)}$ is the standard complex symplectic structure on $\mathcal{CP}(S_k)$, and similarly

$$\omega = (\mathrm{pr}_1)^*\omega^{(1)} + \cdots + (\mathrm{pr}_N)^*\omega^{(N)}\ {}^{3}$$

3. Be wary that in this equality, pr_k stands for the k^{th} projection map $T^*\mathcal{T}(\partial\hat{M}) \to T^*\mathcal{T}(S_k)$ (whereas it stood for the k^{th} projection map $\mathcal{CP}(\partial\hat{M}) \to \mathcal{CP}(S_k)$ in the previous equality). We apologize for these (slightly) misleading notations.

where $\omega^{(k)}$ is the canonical complex symplectic structure on $T^*\mathcal{T}(S_k)$. The equality (5.9) can thus be rewritten:

$$(\mathrm{pr}_1 \circ \tau^\beta)^* \omega^{(1)} + \cdots + (\mathrm{pr}_N \circ \tau^\beta)^* \omega^{(N)} = -i \left[(\mathrm{pr}_1)^* {\omega_G}^{(1)} + \cdots + (\mathrm{pr}_N)^* {\omega_G}^{(N)} \right] .$$

Fix $Z_i \in P(X_i)$ for $i \neq j$ and let us pull back this equality on $\mathcal{CP}(S_j)$ by the map

$$\iota_{(\tilde{Z_i})} : \begin{array}{ccc} \mathcal{CP}(S_j) & \to & \mathcal{CP}(\partial \hat{M}) \\ Z & \mapsto & (Z_1, \ldots, Z_{j-1}, Z, Z_{j+1}, \ldots, X_N) . \end{array}$$

For $k \neq j$, the map $\mathrm{pr}_k \circ \tau^\beta \circ \tilde{\iota}_{(Z_i)}$ maps into the fiber $T_{X_k}{}^*\mathcal{T}(S_k)$, so that $(\tilde{\iota}_{(Z_i)})^* \left((\mathrm{pr}_k \circ \tau^\beta)^* \omega^{(k)} \right) = 0$. Similarly, the map $\mathrm{pr}_k \circ \tilde{\iota}_{(Z_i)}$ maps into the fiber $P(X_k) \subset \mathcal{CP}(S_k)$, so that $(\tilde{\iota}_{(Z_i)})^* \left((\mathrm{pr}_k)^* {\omega_G}^{(k)} \right) = 0$ because p_k is a Lagrangian fibration. For $k = j$, $\mathrm{pr}_k \circ \tau^\beta \circ \tilde{\iota}_{(Z_i)}$ is the map $\tau^\sigma : \mathcal{CP}(S_j) \to T^*\mathcal{T}(S_j)$ and $\mathrm{pr}_k \circ \tilde{\iota}_{(Z_i)}$ is the identity in $\mathcal{CP}(S_j)$. We therefore obtain the desired equality $(\tau^\sigma)^* \omega^{(j)} = -i {\omega_G}^{(j)}$. $\square$

An immediate corollary of this is:

Corollary 5.16. *Generalized Bers sections $\mathcal{T}(S) \to \mathcal{CP}(S)$ are Lagrangian embeddings.*

Another corollary of Theorem 5.15 and Theorem 5.8 is a generalization of McMullen's Theorem 5.9:

Corollary 5.17. *Let $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section. Then*

$$d(\sigma_{\mathcal{F}} - \sigma) = -i\omega_{WP} . \tag{5.11}$$

Finally, we show a generalized version of McMullen's "quasifuchsian reciprocity". To this end, we introduce the notion of "reciprocal generalized Bers embeddings": with the notations of section 1.5, we say that $f : \mathcal{T}(S_j) \to \mathcal{CP}(S_k)$ and $g : \mathcal{T}(S_k) \to \mathcal{CP}(S_j)$ are *reciprocal generalized Bers embeddings* if $f = f_{(X_i)_{i \neq j}, k}$ and $g = f_{(X_i)_{i \neq k}, j}$ for some fixed $X = (X_1, \ldots, X_N) \in \mathcal{T}(\partial \hat{M})$. Since f and g take values in affine spaces, one can consider their differentials:

$$D_{X_j} f : T_{X_j}\mathcal{T}(S_j) \to T^*_{X_k}\mathcal{T}(S_k) \tag{5.12}$$

$$D_{X_k} g : T_{X_k}\mathcal{T}(S_k) \to T^*_{X_j}\mathcal{T}(S_j) \tag{5.13}$$

Theorem 5.18. *Let $f : \mathcal{T}(S_j) \to \mathcal{CP}(S_k)$ and $g : \mathcal{T}(S_k) \to \mathcal{CP}(S_j)$ be reciprocal generalized Bers embeddings as above. Then $D_{X_j} f$ and $D_{X_k} g$ are dual maps. In other words, for any $\mu \in T_{X_j}\mathcal{T}(S_j)$ and $\nu \in T_{X_k}\mathcal{T}(S_k)$,*

$$\langle D_{X_j} f(\mu), \nu \rangle = \langle \mu, D_{X_k} g(\nu) \rangle . \tag{5.14}$$

Proof. The fact that $\beta : \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$ is Lagrangian is written

$$\beta^*\omega_G = (\mathrm{pr}_1 \circ \beta)^*{\omega_G}^{(1)} + \cdots + (\mathrm{pr}_N \circ \beta)^*{\omega_G}^{(N)} = 0 \ ,$$

where ${\omega_G}^{(i)}$ is the standard complex symplectic structure on the component $\mathcal{CP}(S_i)$. Let $\mu \in T_{X_j}\mathcal{T}(S_j)$ and $\nu \in T_{X_k}\mathcal{T}(S_k)$, and define $U, V \in T_X\mathcal{T}(\partial\hat{M})$ by

$$U_i = \begin{cases} 0 & \text{for } i \neq j \\ \mu & \text{for } i = j \end{cases}$$

and

$$V_i = \begin{cases} 0 & \text{for } i \neq k \\ \nu & \text{for } i = k \end{cases} \ .$$

Note that $(\mathrm{pr}_i \circ \beta)_* U$ is vertical in $\mathcal{CP}(S_i)$ whenever $i \neq j$ (resp. $(\mathrm{pr}_i \circ \beta)_* V$ is vertical in $\mathcal{CP}(S_i)$ whenever $i \neq k$). Since the fibers of $\mathcal{CP}(S_i)$ are isotropic (see Theorem 5.11), it follows that $\left((\mathrm{pr}_i \circ \beta)^*{\omega_G}^{(i)}\right)(U,V) = 0$ whenever $i \notin \{j,k\}$. For $i = k$, $(\mathrm{pr}_i \circ \beta)_* U = D_{X_j} f(\mu)$ is still vertical and $(p_k)_* \left((\mathrm{pr}_i \circ \beta)_* V\right) = \nu$ so we can derive from Theorem 5.12 that $\left((\mathrm{pr}_k \circ \beta)^*{\omega_G}^{(k)}\right)(U,V) = i\langle D_{X_j} f(\mu), \nu\rangle$. Similarly, for $i = j$ we have $\left((\mathrm{pr}_j \circ \beta)^*{\omega_G}^{(j)}\right)(U,V) = -\left((\mathrm{pr}_j \circ \beta)^*{\omega_G}^{(j)}\right)(V,U) = -i\langle D_{X_k} g(\nu), \mu\rangle$. In the end, $0 = (\beta^*\omega_G)(U,V) = i\langle D_{X_j} f(\mu), \nu\rangle - i\langle D_{X_k} g(\nu), \mu\rangle$ as desired. □

We would like to emphasize that Corollary 5.17 is not an immediate consequence of "Kleinian reciprocity": the proof requires Lagrangian information. On the other hand, Steven Kerckhoff pointed out to us that Theorem 5.18 can be derived from Kleinian reciprocity without using our previous results, rightly so (the proof is easily adapted taking ω^β instead of ω_G, avoiding the use of the symplectic structure on $\mathcal{CP}(S)$).

5.3 Darboux coordinates

It is an immediate consequence of the analytical continuation property 5.7 and Wolpert's theorem 4.7 that complex Fenchel-Nielsen coordinates are Darboux coordinates for the standard symplectic structure on the quasifuchsian space:

Theorem 5.19. *Let α be any pants decomposition of S. Complex Fenchel-Nielsen coordinates $(l_\alpha^{\mathbb{C}}, \beta_\alpha^{\mathbb{C}})$ on the quasifuchsian space $\mathcal{QF}(S)$ are Darboux coordinates for the standard complex symplectic structure:*

$$\omega_G = \sum_{i=1}^{N} dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}} \ . \tag{5.15}$$

Proof. Of course, Theorem 5.7 is still true when replacing $\mathcal{CP}(S)$ by any connected neighborhood of the Fuchsian slice $\mathcal{F}(S)$, such as the quasifuchsian space $\mathcal{QF}(S)$. Let $\omega = \sum_{i=1}^{N} dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}}$. Since $l_{\alpha_i}^{\mathbb{C}}$ and $\tau_{\alpha_i}^{\mathbb{C}}$ are holomorphic, ω is a complex symplectic structure on $\mathcal{QF}(S)$. In restriction to the Fuchsian slice, $\iota^*\omega = \sum_{i=1}^{N} dl_{\alpha_i} \wedge d\tau_{\alpha_i}$ where (l_α, τ_α) are the classical Fenchel-Nielsen coordinates. By Wolpert's Theorem 4.7, it follows that $(\sigma^{\mathcal{F}})^*\omega = \omega_{WP}$. This proves that $\omega = \omega_G$ according to Theorem 5.7. $\square$

Of course, this shows in particular:

Corollary 5.20. *Platis' symplectic structure ω_P is equal to the standard complex symplectic structure ω_G in restriction to the quasifuchsian space $\mathcal{QF}(S)$.*

Notice how the analytic continuation argument provides a very simple alternative proof of Platis' result that the symplectic structure $\sum_{i=1}^{N} dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}}$ does not depend on a choice of a pants decomposition (relying, of course, on Wolpert's result).

In [Gol04], Goldman gives a fairly extensive description of the complex symplectic structure ω_G on the character variety $\mathcal{X}(S, SL_2(\mathbb{C}))$, discussing in particular the "Hamiltonian picture". We recover that the Hamiltonian flow of a complex length function is the associated complex twist flow.

5.4 Minimal surfaces and renormalized volume

Let us now address the symplectic structure $\alpha^*\omega$ defined by the "minimal surface identification" α as in section 2.4. This will be done by introducing the notion of renormalized volume of almost-Fuchsian manifolds. We refer to [KS08] and [KS] for a systematic presentation of the renormalized volume of hyperbolic 3-manifolds.

Equidistant foliations in quasifuchsian 3-manifolds

Let M be a quasifuchsian 3-manifold. Consider a smooth embedded surface S_0 such that the normal exponential map induces an isotopic deformation of S_0 on the ideal boundary component $\partial_\infty^+ M$. In particular, S_0 disconnects M (or equivalently $\hat{M} = M \cup \partial_\infty M$) into two connected components, each one containing a boundary component. Denote by $[S_0, \partial_\infty^+ M]$ (resp. $[\partial_\infty^- M, S_0]$) the connected component of $\hat{M}$ containing $\partial_\infty^+ M$ (resp. $\partial_\infty^- M$). Let S_ρ be the set of points at distance $\rho \geqslant 0$ of S_0 in $[S_0, \partial_\infty^+ M]$. We assume now that S_0 is convex, in the sense that $]\partial_\infty^- M, S_0]$ is geodesically convex in M. The nearest-point projection $\kappa : [S_0, \partial_\infty^+ M[\to S_0$ is well-defined and smooth, and it admits a natural extension to $\partial_\infty^+ M$. It is easy to see that S_ρ is a smooth embedded surface, obtained as the image of a section $u_\rho : S_0 \to S_\rho$

to κ [4] (including in the case $\rho = +\infty$, with $S_\infty = \partial_\infty^+ M$). The 1-parameter family $(S_\rho)_{\rho \geqslant 0}$ is called an equidistant foliation of the topological end $\partial^+ \hat{M}$, it is a smooth foliation of $[S_0, \partial_\infty^+ M]$ by equidistant surfaces. Two equidistant foliations $(S_\rho)_{\rho \geqslant 0}$ and $(S'_\rho)_{\rho \geqslant 0}$ are considered equivalent if $S_0 = S'_\rho$ for some $\rho > 0$ or vice-versa.

Let $(S_\rho)_{\rho \geqslant 0}$ be an equidistant foliation as above. For any $\rho \geqslant 0$, the map $f_\rho = u_\rho \circ {u_\infty}^{-1}$ is a smooth identification $f_\rho : S_\infty \xrightarrow{\sim} S_\rho$. Let $I_\rho := {f_\rho}^* I_{S_\rho}$ where I_{S_ρ} is the first fundamental form on S_ρ, and define II_ρ and III_ρ similarly. Writing down the differential equation satisfied by I_ρ and integrating it, one shows (see [KS07]):

Proposition 5.21. *For any $\rho, h > 0$,*

$$I_{\rho+h} = (\cosh^2 h)\, I_\rho + 2(\sinh h \cosh h)\, II_\rho + (\sinh^2 h)\, III_\rho \quad . \tag{5.16}$$

The following easily follows:

Proposition 5.22.
- *$I_\rho \sim_{\rho \to \infty} \frac{e^{2\rho}}{2} I^*$, where $I^* := \frac{1}{2}(I_0 + 2\, II_0 + III_0)$.*
- *For any $\rho \geqslant 0$, $I^* = \frac{e^{-2\rho}}{2}(I_\rho + 2\, II_\rho + III_\rho)$.*
- *$II^* := \frac{1}{2}(I_\rho - III_\rho)$ does not depend on ρ.*

A standard argument shows that I^* must belong to the conformal class of the ideal boundary $\partial_\infty^+ M$. Furthermore, a theorem first proved by C. Epstein (see also [HR93]) says that:

Theorem 5.23 (C. Epstein)**.** *Given any metric g in the conformal class of $\partial_\infty^+ M$, there is a unique equidistant foliation up to equivalence $(S_\rho)_{\rho \geqslant 0}$ such that $I^* = g$.*

Renormalized volume of almost-Fuchsian manifolds

Suppose now that M is almost-Fuchsian, so that it contains a unique minimal surface Σ. Let $(S_\rho)_{\rho \geqslant 0}$ be the unique equidistant foliation associated to the Poincaré metric on $\partial_\infty^+ M$. Denote by $N_\rho = [\Sigma, S_\rho]$ be the compact connected component of $M \setminus (\Sigma \cup S_\rho)$ and by $V(N_\rho)$ its hyperbolic volume. A direct calculation shows that:

Proposition 5.24. *The number*

$$W = V(N_\rho) - \frac{1}{4}\left(\int_{S_\infty} H_\rho da_\rho\right) - 2\pi(g-1)\rho \tag{5.17}$$

does not depend on ρ. W will be called the renormalized volume *of the almost-Fuchsian manifold M.*

4. Note that κ is a right inverse to the normal exponential map, and u_ρ is just the time-ρ normal exponential map.

In this definition, H_ρ is of course the mean curvature $H_\rho = \mathrm{tr}(I_\rho^{-1} \mathit{II}_\rho)$ and da_ρ is the area element for the metric I_ρ.

Note that the renormalized volume actually defines a function $W : \mathcal{AF}(S) \to \mathbb{R}$. Next we want to compute its derivative dW. In other words, how is W affected by an infinitesimal deformation in $\mathcal{AF}(S)$? First, a formula for the variation of volume was proved by Rivin-Schlenker [RS99]:

Theorem 5.25 (Rivin-Schlenker [RS99]). *Let M be a convex cocompact hyperbolic 3-manifold and let $N \subset M$ be a convex compact subset of M with smooth boundary. Under an infinitesimal deformation of the hyperbolic metric on M,*

$$2\delta V(N) = \int_{\partial N} \left(\delta H + \frac{1}{2} \langle \delta\, I, \mathit{II} \rangle_I \right) da_I \ . \tag{5.18}$$

We use this to show:

Theorem 5.26. *Let M be an almost-Fuchsian manifold. Under an infinitesimal deformation of the hyperbolic metric on M, the variation of the renormalized volume is given by*

$$\delta W = -\frac{1}{4} \int_{S_\infty} \langle \mathit{II}_\Sigma, \delta\, I_\Sigma \rangle \, da_\Sigma - \frac{1}{4} \int_{S_\infty} \langle \mathit{II}_0^*, \delta\, I^* \rangle \, da^* \tag{5.19}$$

where $\mathit{II}_0^* = \mathit{II}^*$ is the trace-free part of II^*.

Proof. We only give an outline of the proof, most of the calculations needed have already been done in [KS08]. Following the definition of W, one has

$$\delta W = \delta V(N_\rho) - \frac{1}{4} \left(\int_{S_\infty} \delta H_\rho da_\rho \right) - \frac{1}{4} \left(\int_{S_\infty} H_\rho \delta(da_\rho) \right)$$

Using the formula 5.25, the variation of $V(N_\rho)$ is given by

$$\delta V(N_\rho) = -\frac{1}{4} \left(\int_{S_\infty} \langle \delta\, I_\Sigma, \mathit{II}_\Sigma \rangle_{I_\Sigma} \right) da_\Sigma + \frac{1}{2} \left(\int_{S_\infty} \delta H_\rho + \frac{1}{2} \langle \delta\, I_\rho, \mathit{II}_\rho \rangle_{I_\rho} da_\rho \right) \ .$$

The variation of the mean curvature H on a surface is $\delta H = -\langle \delta\, I, \mathit{II} \rangle_I + \langle I, \delta\, \mathit{II} \rangle_I$, and the variation of the area element is given by $\delta(da) = \frac{1}{2} \langle \delta\, I, I \rangle_I \, da$. Putting all this together, one finds

$$\delta W = -\frac{1}{4} \left(\int_{S_\infty} \langle \delta\, I_\Sigma, \mathit{II}_\Sigma \rangle_{I_\Sigma} \right) da_\Sigma + \frac{1}{4} \left(\int_{S_\infty} \left\langle \delta\, \mathit{II}_\rho - \frac{H_\rho}{2} \delta\, I_\rho, I_\rho \right\rangle_{I_\rho} da_\rho \right) \ .$$

A tedious but straightforward calculation done in [KS08] (Corollary 6.2) shows that for any foliation $(S_\rho)_{\rho \geqslant 0}$,

$$\int_{S_\infty} \left\langle \delta\, \mathit{II}_\rho - \frac{H_\rho}{2} \delta\, I_\rho, I_\rho \right\rangle_{I_\rho} da_\rho = -\int_{S_\infty} \delta H^* + \langle \delta\, I^*, \mathit{II}_0^* \rangle_{I^*} \, da^* \ .$$

It is also shown there (Remark 5.4) that $H^* = -K^*$. However, we have chosen I^* such that $K^* = -1$, it follows that $\delta H^* = 0$. $\square$

We need this last ingredient, proven in [KS08]:

Proposition 5.27 (Krasnov-Schlenker [KS08])**.** *Let $M = M_Z$ be a quasifuchsian manifold. Let $(S_\rho)_{\rho \geqslant 0}$ be the unique equidistant foliation associated to the Poincaré metric on $\partial^+_\infty M$ and let II^*_0 $(= II^* - I^*)$ be the trace-free part of II^* as above. Then*

$$II^*_0 = \mathrm{Re}(\tau^{\sigma_\mathcal{F}}(Z)) \ . \tag{5.20}$$

As a consequence of this and the previous theorem, we obtain:

Theorem 5.28. *Let $W : \mathcal{AF}(S) \to \mathbb{R}$ be the renormalized volume function on the almost-Fuchsian space, α be the minimal surface identification as in section 2.4. Then*

$$dW = -\frac{1}{4}\mathrm{Re}\left[\alpha^*\xi + (\tau^{\sigma_\mathcal{F}})^*\xi\right] \tag{5.21}$$

where ξ is the canonical one-form on the complex cotangent space $T^*\mathcal{T}(S)$ (see section 2.2).

The following identification easily follows, with (5.8):

Corollary 5.29.

$$\mathrm{Re}(\alpha^*\omega) = -\mathrm{Im}(\omega_G) \ . \tag{5.22}$$

CHAPTER 6

The hyperkähler structure

6.1 Hyperkähler manifolds

A hyperkähler manifold is defined as a Riemannian manifold (M, g) equipped with three orthogonal complex structures I, J and K that are parallel (for the Levi-Civita connection) and satisfy the quaternion algebra identities:

$$\begin{aligned} &I^2 = J^2 = K^2 = -\mathrm{id}_{TM} \\ &IJ = -JI = K \ . \end{aligned}$$

Note that each one of these complex structures turns (M, g) into a Kähler manifold. In fact, a hyperkähler manifold admits a Euclidean sphere of Kähler structures: for any $u = (b, c, d) \in S^2 \subset \mathbb{R}^3$ [1], the operator $I_u := bI + cJ + dK$ is an integrable orthogonal complex structure on (M, g). The triplet (I, J, K) does not play a distinguished role among triplets (I_u, I_v, I_w) that are associated to direct orthonormal bases (u, v, w) of $\mathbb{R}^3$. Hyperkähler structures are extremely strong (hence "rare") structures [2].

We denote by ω_{I_u} the Kähler form associated to the complex structure I_u. It is clear that only three non-redundant structures among g, I, J, K, ω_I, ω_J, ω_K are

1. Here S^2 is the standard unit sphere in $\mathbb{R}^3$.

2. The quaternionic vector space $\mathbb{H}^n$ with the flat metric is obviously an example of a hyperkähler manifold (but the quaternionic projective space $\mathbb{H}\mathbf{P}^n$ is not, although it is *quaternion-Kähler*, see e.g. [Bes08]). However, the hyperkähler metric found by Eguchi-Hanson in [EH79] on the cotangent bundle of $\mathbb{C}\mathbf{P}^1$ was the first nontrivial complete example. Calabi generalized this example in every dimension in [Cal79] and coined the name "hyperkähler". Note that the holonomy of a $4n$-dimensional hyperkähler manifold lies in the group $Sp(n)$, which is the intersection of all the holonomy groups of $4n$-dimensional manifolds in Berger's list.

necessary to completely determine the hyperkähler structure [3]. We say that three such structures generate the hyperkähler structure.

There are several ways to see that a hyperkähler structure is a refinement of a complex-symplectic structure [4]. For example, it is easy to check that the complex-valued 2-form $\omega_J + i\omega_K$ is a complex symplectic form on (M, I).

6.2 Complexification of a real analytic Kähler manifold

We have seen the notion of complexification of a real analytic manifold X in section 5.1. It is not hard to show that such a complexification always exists and that its *germ* $X^{\mathbb{C}}$ is unique [5]. Developing the elementary arguments exposed there, one shows that real analytic tensors on X admit unique holomorphic extensions to $X^{\mathbb{C}}$. For example, if X is equipped with a real analytic symplectic form ω, there is a unique complex symplectic structure that extends ω on $X^{\mathbb{C}}$. We showed in the previous chapter that the complex symplectic structures ω_G, ω_P and $-i\omega^\sigma$ (where σ is a generalized Bers section) are all equal to the complexification of the Weil-Petersson Kähler form on the Fuchsian slice.

Since a Kähler manifold X has additional structure, it is natural to ask what this structure yields on the complexification $X^{\mathbb{C}}$. Well, it is quite clear that

- The complexification of the Kähler form ω is a complex symplectic form (as mentioned above).
- The complexification of the Riemannian metric g is a holomorphic metric $g^{\mathbb{C}}$ [6].
- The compatibility of the complex structure on X is expressed in the complexification by the fact that $T^{1,0}X^{\mathbb{C}}$ and $T^{0,1}X^{\mathbb{C}}$ are isotropic for $g^{\mathbb{C}}$.
- The integrability of the complex structure on X is expressed by the fact that the complexification of the Levi-Civita connection is a holomorphic connection on $X^{\mathbb{C}}$ that preserves the splitting $T^{1,0}X^{\mathbb{C}} \oplus T^{0,1}X^{\mathbb{C}}$.

However, it is unclear to the author whether there is (a candidate of) a natural hyperkähler structure on the complexification $X^{\mathbb{C}}$ that combines these structures. Furthermore, in our situation of interest where $X = \mathcal{F}(S)$ and $X^{\mathbb{C}} = \mathcal{QF}(S)$, there is even more additional structure at disposal [7]. It is thus legitimate to ask:

Question 1. Is there a (unique) natural hyperkähler structure on some neighborhood of the Fuchsian slice in $\mathcal{CP}(S)$?

3. For example, g, I and ω_I are obviously redundant, but also I, ω_J and ω_K (check that $\omega_K^\flat \circ I = \omega_J^\flat$).
4. This shows in particular that a hyperkähler manifold is Ricci-flat.
5. This means that two complexifiations of X agree on some neighborhood of X.
6. *i.e.* a holomorphic symmetric complex-bilinear form.
7. For instance, $\mathcal{QF}(S)$ admits two transverse holomorphic Lagrangian foliations whose leaves carry affine structures.

Of course, defining “natural” here is part of the question. For instance, we would like the hyperkähler structure to refine the complex symplectic structure ω_G and to extend the Weil-Petersson metric off the Fuchsian slice.

6.3 The cotangent hyperkähler structure

Feix (in [Fei01]) and Kaledin (in [Kal99]) showed independently that there is a unique canonical hyperkähler structure on the total space of the cotangent bundle of any real analytic Kähler manifold, defined in some neighborhood of the zero section. More precisely:

Theorem 6.1 (Feix ([Fei01]), Kaledin [Kal99])**.** *Let X be a real analytic Kähler manifold. There exists a unique hyperkähler metric in a neighborhood of the zero section in T^*X such that:*
- *The hyperkähler structure refines the canonical complex symplectic structure.*
- *The hyperkähler metric extends the Riemannian metric off the zero section.*
- *The $U(1)$ action in the fibers of T^*X (by multiplication) is isometric.*

Of course, it is only natural to ask how this hyperkähler structure relates to the structures that exist on the complexification $X^{\mathbb{C}}$. There might be a very simple answer to this question but it escapes the author’s limited understanding.

As we have seen in section 2.3, any choice of a section $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ yields an affine isomorphism $\tau^\sigma : \mathcal{CP}(S) \xrightarrow{\sim} T^*\mathcal{T}(S)$. It follows that any such choice defines a unique “cotangent” hyperkähler structure in a neighborhood of the slice $\sigma(\mathcal{T}(S))$. It is a direct consequence of our Theorem 5.15 that:

Theorem 6.2. *If σ is any generalized Bers section*[8]*, the cotangent hyperkähler structure as above refines the complex symplectic structure $-i\omega_G$.*

The following questions appear to be legitimate:

Question 2.
- Are the cotangent hyperkähler structures as above defined on the whole quasi-fuchsian space?
- How do the cotangent hyperkähler structures provided by different choices of σ compare, especially if σ is elected among Bers sections?
- How do these hyperkähler structures compare to the one mentioned in question 1 (provided it exists)?

In trying to answer these questions, the results obtained by Donaldson in [Don03] and T.Hodge in [Hod05] should prove to be very useful. In [Don03], Donaldson

8. or more generally if $d(\sigma_{\mathcal{F}} - \sigma) = -i\omega_{WP}$, see Theorem 5.8.

explicited a construction of a hyperkähler extension of $\mathcal{T}(S)$ using the notion of hyperkähler (or "trisymplectic") reduction. In [Hod05], T.Hodge shows that this hyperkähler structure refines the complex symplectic structure ω_G and agrees with the Feix-Kaledin hyperkähler structure obtained on a neighborhood of the Fuchsian slice under the minimal surface identification α (see section 2.10). Our theorem 5.29 might also provide some insight, in particular:

Question 3. Do the three symplectic structures $\mathrm{Re}(\omega_G)$, $\mathrm{Im}(\omega_G)$ and $\mathrm{Im}(\alpha^*\omega)$ generate a hyperkähler structure on the almost Fuchsian space $\mathcal{AF}(S)$? If so, how does it compare to the possible hyperkähler structure mentioned in question 1?

6.4 Higgs bundles

We will be very brief in this last section. The theory of Higgs bundles developed by N. Hitchin in [Hit87] based on the so-called self-duality equations on a Riemann surface provides a hyperkähler structure on the smooth part of the character variety $\mathcal{X}(S)$, after some some identifications between different deformation theories [9]. The deformation space $\mathcal{CP}(S)$ thus inherits a hyperkähler structure through holonomy. However, the whole construction depends on the choice of a conformal structure on S. Of course, the main question we would like to address is:

Question 4. How does the Hitchin hyperkähler structure compare to the hyperkähler structures previously mentioned?

It is noteworthy that Hitchin's construction actually provides a family of hyperkähler structures parametered by Teichmüller space, just like the cotangent Feix-Kaledin hyperkähler structures obtained by the choices of Bers sections. What is the relation between these two families of hyperkähler structures? Also, we observe that Hitchin constructs the hyperkähler metric as the reduction of a flat hyperkähler structure on an infinite-dimensional space, just like Donaldson in [Don03], maybe suggesting connections between their respective theories.

Answering all the questions asked in this concluding chapter would give a satisfactory, unifying description of the hyperkähler structure of the deformation space $\mathcal{CP}(S)$. However, it is likely that this would only be the start of a perhaps more geometric investigation of this hyperkähler structure, hopefully providing more insight into complex projective geometry.

9. Mainly, the moduli space of flat reductive principal connections on a Riemann surface X and the moduli space of polystable Higgs bundles on X, see e.g. [BGPG06].

Notation Index

Latin notations

Ad	p.36	adjoint representation $G \to \mathrm{Aut}\mathfrak{g}$
$\mathcal{AF}(S)$	33	almost-Fuchsian deformation space
B	31	shape operator
B	36	Killing form on $\mathfrak{g}$
$\tilde{B}$	36	$= \frac{1}{4}B$, “trace form” on $\mathfrak{sl}_2(\mathbb{C})$
$\mathcal{CP}(S)$	19	deformation space of complex projective structures on S
$\partial_\infty M$	23	ideal boundary of M
$\mathcal{F}(S)$	22	Fuchsian (or Fricke p.38) deformation space of S
f_{X^+}	23	Bers embedding $\mathcal{T}(\bar{S}) \to P(X)$
$f_{(X_i),k}$	25	generalized Bers embedding $\mathcal{T}(S_j) \to P(X_k)$
g	19	genus of S
$\mathfrak{g}$	35	Lie algebra of G
G	35	Lie group (mainly $PSL_2(\mathbb{C})$)
hol	37	holonomy map
H	31	mean curvature
$\mathcal{HC}(M)$	24	deformation space of convex cocompact hyperbolic metrics
J	51	almost complex structure
K	31	Gaussian curvature
l_γ	44	hyperbolic length function
$l_\gamma^{\mathbb{C}}$	49	complex length function
M	23	hyperbolic 3-manifold
$\hat{M}$	23	compact 3-manifold whose interior is M
p	21	forgetful projection $\mathcal{CP}(S) \to \mathcal{T}(S)$
$P(X)$	28	$= p^{-1}(\{X\})$
pr_k	24	k^{th} projection map
$Q(X)$	20	space of holomorphic quadratic differentials on X
$\mathcal{QF}(S)$	23	quasifuchsian deformation space of S
$\mathcal{R}(S)$	35	$= \mathcal{R}(S, G) = \mathrm{Hom}(\pi_1(S), G)$
s_0	29	zero section to $\pi : T^*\mathcal{T}(S) \to \mathcal{T}(S)$
$\mathfrak{sl}_2(\mathbb{C})$	35	Lie algebra of $PSL_2(\mathbb{C})$

S	19	closed oriented surface of genus $g \geqslant 2$
$\overline{S}$	22	S with reversed orientation
$\mathcal{S}f$	27	Schwarzian derivative of f
tw_γ	45	twist flow
$\mathrm{tw}_\gamma^{\mathbb{C}}$	49	complex twist flow
TM	20	tangent bundle of a manifold M
$T^*\mathcal{T}(S)$	29	holomorphic cotangent bundle of $\mathcal{T}(S)$
$\mathcal{T}(S)$	19	Teichmüller space of S
X	19	generic complex structure on S
X_f	46	Hamiltonian vector field of f
$\mathcal{X}(S)$	35	$= \mathcal{X}(S, G)$, character variety
Z	19	generic complex projective structure on S

Greek notations

α	p.43	generic pants decomposition of S
α	33	minimal surface identification
β	25	simultaneous uniformization section to $p : \mathcal{CP}(\partial\hat{M}) \to \mathcal{T}(\partial\hat{M})$
γ	43	generic element of $\pi_1(S)$
γ^X	43	geodesic representative of γ
Γ	23	generic Kleinian group
$\iota_{(X_i)}$	25	canonical injection $\mathcal{T}(S_j) \to \mathcal{T}(\partial\hat{M})$
μ	20	generic Beltrami differential
ξ	29	canonical 1-form on $T^*\mathcal{T}(S)$
π	29	canonical projection $T^*\mathcal{T}(S) \to \mathcal{T}(S)$
π	35	$= \pi_1(S)$
ρ	35	generic representation $\in \mathcal{R}(S)$
σ	28	generic section to $p : \mathcal{CP}(S) \to \mathcal{T}(S)$
σ_{X^-}	23	Bers section
$\sigma_{(X_i)}$	25	generalized Bers section
$\sigma_{\mathcal{F}}$	22	Fuchsian section
Σ	32	minimal surface
τ_γ	45	twist parameter
$\tau_\gamma^{\mathbb{C}}$	49	complex twist parameter
τ^σ	28	affine isomorphism $\mathcal{CP}(S) \xrightarrow{\sim} T^*\mathcal{T}(S)$ such that $\tau^\sigma \circ \sigma = s_0$
φ	20	generic holomorphic quadratic differential
ω	29	canonical complex symplectic form on $T^*\mathcal{T}(S)$
ω_G	37	Goldman's symplectic form on $\mathcal{X}(S)$ or $\mathcal{CP}(S)$
ω_P	47	Platis' symplectic form on $\mathcal{QF}(S)$
ω^σ	29	$= (\tau^\sigma)^*\omega$
ω_{WP}	21	Weil-Petersson Kähler form

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