Harmonic maps from Kähler manifolds

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Abstract

We review the theory of harmonic and pluriharmonic maps from complex and Kähler manifolds to Riemannian manifolds. We explain applications to rigidity results when the target manifold is nonpositively curved in some strong sense, a condition met by symmetric spaces of noncompact type, and the importance thereof for the nonabelian Hodge correspondence. This report was written following an expository talk that the author gave at the workshop *Harmonic maps and rigidity* that took place in Sisteron, France, 6-14 April 2019.

Key words and phrases: Harmonic maps · Pluriharmonic maps · Kähler manifolds · Symmetric spaces · Bochner formula · Rigidity · Higgs bundles

2000 Mathematics Subject Classification: Primary: 58E20; Secondary: 53C43 · 53C55 · 32W50 · 53C24

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Introduction

The theory of harmonic maps between Riemannian manifolds was properly started by the foundational paper of Eells-Sampson [ES64]. A smooth map $f\colon M \to N$ is *harmonic* if it is a critical point of the energy functional

$$E(f) = \frac{1}{2}\int_M \|\mathrm{d}f\|^2 \,\mathrm{d}v\ .$$

Equivalently, f solves the corresponding Euler-Lagrange equation, which is the nonlinear PDE

$$\Delta f = 0$$

where the Laplacian Δ is a nonlinear operator (more commonly denoted $\tau(f)$ and called *tension field*) generalizing the Riemannian Laplacian. We recall the definition of this operator in § 1 and explain why $\Delta f = 0$ is equivalent to $\mathrm{d}f$ being a harmonic one-form in the sense of Hodge theory.

The PDE $\Delta f = 0$ is typically determined and Eells-Sampson proved the existence of harmonic maps when the target manifold N has nonpositive sectional curvature (and, for example, both M and N are compact), using a heat flow technique. The success of this technique relies on a Bochner formula for maps between Riemannian manifolds generalizing the classical Bochner-Weitzenböck formulas. They additionally showed rigidity results under stronger curvature assumptions, such as nonnegative Ricci curvature for the domain manifold and negative sectional curvature for the target manifold. These rigidity results again rely entirely on the Bochner formula.

In the seminal paper [Siu80], Siu adapted Eells-Sampson's Bochner techniques to the setting where the domain manifold M is Kähler. Siu's ideas were further developed by Sampson [Sam86] and Carlson-Toledo [CT89]. Sampson showed that if M is compact Kähler and N has nonpositive Hermitian sectional curvature, then any harmonic map $M \to N$ is actually *pluriharmonic*, a stronger notion than harmonicity. Moreover Siu's Bochner technique yields a constraint on the relation between the image of $\mathrm{d}f$ in $\mathrm{T}N$ and the curvature tensor of N, leading to strong rigidity results under the appropriate curvature assumptions. In particular, there are remarkable consequences when $N = G/K$ is a symmetric space of noncompact type. We will also briefly discuss the significance of these results for the theory of Higgs bundles and the nonabelian Hodge correspondence initiated by Hitchin [Hit87], Donaldson [Don87], Corlette [Cor88], and Simpson [Sim91].

The goal of this report is to give a clean presentation of these notions and results, after developing the relevant mathematical background mostly consisting of standard differential geometry.

Acknowledgments

The author wishes to deeply thank the workshop organizers Alexis Gilles, Nicolas Tholozan, and Jérémy Toulisse, for organizing an absolutely wonderful workshop *Harmonic maps and rigidity*, which was a complete success. He also thanks all the workshop participants for very stimulating mathematical discussions. For more details on the workshop, please visit the dedicated web page: https://sites.google.com/view/jtoulisse/accueil/sisteron-2019

The author gratefully acknowledges research support from the NSF Grant DMS1107367 aka *RNMS: GEometric structures And Representation varieties* (the *GEAR Network*). He also gratefully acknowledges research support from the AG Geometrie of the Mathematics Department of TU Darmstadt.

1 Harmonic maps

Let M and N be Riemannian manifolds and $f\colon M \to N$ a smooth map.

The main purpose of this section is to explain the equivalence:

$$f \text{ harmonic} \quad \Leftrightarrow \quad \Delta f = 0 \quad \Leftrightarrow \quad \mathrm{d}_\nabla^* \mathrm{d} f = 0 \quad \Leftrightarrow \quad \mathrm{d} f \text{ is a harmonic 1-form}$$

(assuming M is compact for the last one). This requires introducing the basic tools and notions of differential geometry at play in the theory of harmonic maps between Riemannian manifolds. There are quite a few good references on the subject, such as [ES64], [EL83], [EL95b], [Jos08], [Xin96], [Chi13], [Jos17, Chap. 9]. Please refer to these for examples and more developments. Note that contrary to many authors, we do not feel the need to write any formulas in local coordinates.

Remark 1.1. In all this report, M is assumed connected and orientable, and given a fixed orientation. In particular, the volume density v_M on M can be identified to the volume form vol_M.

1.1 f is harmonic if and only if $\Delta f = 0$

Let us introduce some notations: ∇^M and ∇^N are the Levi-Civita connections of M and N respectively. Consider the pullback vector bundle

$$E := f^*(\mathrm{T}N) \to M$$

whose fiber above $x \in M$ is $\mathrm{T}_{f(x)}N$. E admits a pullback connection $f^*\nabla^N$ uniquely determined by the relation $(f^*\nabla^N)_X f^*s = f^*(\nabla^N_X s)$ for any section $s \in \Gamma(N, \mathrm{T}N)$. From now on we shall denote this connection $\nabla := f^*\nabla^N$.

Consider the vector bundle $\mathrm{T}^*M \otimes E \to M$. It admits a tensor product connection $\bar{\nabla}$ induced from the dual connection of ∇^M in T^*M and the connection ∇ in E. Thus $\bar{\nabla}$ is a linear map:

$$\Gamma(\mathrm{T}^*M \otimes E) \to \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M \otimes E)\ .$$

(We always denote Γ the space of smooth sections.)

Note that $\mathrm{d}f$, which is a linear map $\mathrm{T}M \to \mathrm{T}N$ (or rather a morphism of vector bundles), can be seen as a section of $\mathrm{T}^*M \otimes E$. It therefore makes sense to consider $\bar{\nabla}(\mathrm{d}f)$, it is an element of $\Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M \otimes E)$. In other words it is a bilinear map $\mathrm{T}M \times \mathrm{T}M \to E$. By definition, this is the *Hessian* of f. Abusing notations, we denote it $\nabla^2 f := \bar{\nabla}(\mathrm{d}f)$. The justification of this notation is that it is reasonable to write ∇f instead of $\mathrm{d}f$ (for any connection ∇ on M, $\mathrm{d}f$ and ∇f coincide for real-valued functions), and if we also abusively write ∇ instead of $\bar{\nabla}$ for the connection in $\mathrm{T}^*M \otimes E$, we indeed have $\nabla^2 f = \nabla\nabla f = \bar{\nabla}(\mathrm{d}f)$.

Proposition 1.2. *The Hessian of f is symmetric as a bilinear map*

$$\nabla^2 f\colon \mathrm{T}M \times \mathrm{T}M \to \mathrm{T}N\ .$$

Proof. This is essentially a consequence of the fact that ∇^M and ∇^N are torsion-free. Let u, v be tangent vectors at some point $x \in M$, let us show that $\nabla^2 f(u, v) - \nabla^2 f(v, u) = 0$. By definition of the product connection $\bar{\nabla}$,

$$\nabla_u(\mathrm{d}f(v)) = (\bar{\nabla}_u(\mathrm{d}f))(v) + \mathrm{d}f(\nabla^M_u v)\ . \tag{1}$$

Note that for (1) to make sense, we locally extend v as a vector field around x. Let us rewrite (1) as:

$$\nabla^2 f(u,v) = \nabla^N_{\mathrm{d}f(u)}(\mathrm{d}f(v)) - \mathrm{d}f(\nabla^M_u v) .$$

Thus we can write

$$\nabla^2 f(u,v) - \nabla^2 f(v,u) = \left[\nabla^N_{\mathrm{d}f(u)}(\mathrm{d}f(v)) - \nabla^N_{\mathrm{d}f(v)}(\mathrm{d}f(u))\right] - \left[\mathrm{d}f(\nabla^M_u v) - \mathrm{d}f(\nabla^M_v u)\right] . \quad (2)$$

Since ∇^M and ∇^N are both torsion-free, (2) is rewritten

$$\nabla^2 f(u,v) - \nabla^2 f(v,u) = [\mathrm{d}f(u), \mathrm{d}f(v)] - \mathrm{d}f([u,v]) .$$

This is zero by naturality of the Lie backet. □

By definition, the *Laplacian* Δf is the trace of the Hessian $\nabla^2 f$. Recall that in the presence of an inner product, one can take the trace of any symmetric bilinear form: it is the trace of the associated self-adjoint endomorphism. Alternatively, it is the trace of the matrix associated to the bilinear form in any orthonormal basis. Let us record these definitions:

Definition 1.3. The *Hessian* of f is the symmetric bilinear map

$$\nabla^2 f \colon \mathrm{T}M \times \mathrm{T}M \longrightarrow \mathrm{T}N$$

defined by $\nabla^2 f = \bar{\nabla}(\mathrm{d}f)$. The *Laplacian* Δf is the section of $f^*(\mathrm{T}N)$ defined by

$$\Delta f = \mathrm{tr}(\nabla^2 f) .$$

Remark 1.4. The Hessian $\nabla^2 f$ generalizes both the Riemannian Hessian (for real-valued functions) and the second fundamental form (for isometric immersions); it is sometimes called the *(vector-valued) second fundamental form* of f. The Laplacian Δf is called *tension field* by most authors and denoted $\tau(f)$, probably by herd behavior after the foundational paper of Eells-Sampson [ES64].

For our purposes, it is good enough to define a harmonic map as a smooth map $f \colon M \longrightarrow N$ such that $\Delta f = 0$: the reader may take this as a definition and skip the remainder of this subsection. But for completeness, we recall why $\Delta f = 0$ is equivalent to f being a critical point of the energy functional. Essentially, this is because $\mathrm{grad}\, E(f) = -\Delta f$ (Remark 1.7).

Let $f \colon M \longrightarrow N$ be a smooth map. If M is not compact, the energy functional $E(f)$ can be infinite, so instead we work on compact subsets $K \subseteq M$:

$$E_K(f) \coloneqq \frac{1}{2}\int_K \|\,\mathrm{d}f\|^2 \,\mathrm{d}v_M . \quad (3)$$

In (3), $\|\,\mathrm{d}f\|$ is the *Hilbert-Schmidt norm* of f: Given any linear map $L \colon V \longrightarrow W$ between Euclidean vector spaces $(V, g_V = \langle\cdot,\cdot\rangle_V)$ and $(W, g_W = \langle\cdot,\cdot\rangle_V)$, the Hilbert-Schmidt norm of L is defined by $\|L\|^2 = \mathrm{tr}_{g_V}(L^* g_W)$. More abstractly, it is the norm of L with respect to the inner product $g_V^* \otimes g_W$. More concretely, $\|L\|^2 = \mathrm{tr}({}^t M M)$ where M is the matrix of L taken with respect to any pair of orthonormal bases of V and W.

Now let V be any infinitesimal variation of f, i.e. $V \in \Gamma(f^*\,\mathrm{T}N)$, and let (f_t) be any 1-parameter variation of f with initial tangent vector V, i.e. $(f_t) \colon I \times M \longrightarrow N$ is smooth (where $I \subseteq \mathbb{R}$ is an interval containing 0), $f_0 = f$, and $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f_t = V$. One says that this variation is supported the compact set K if $f_t = f$ outside of K.

Theorem 1.5 (First variational formula for the energy)**.** *For any variation (f_t) of f supported in a compact set K and with initial tangent vector V,*

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} E_K(f_t) = -\int_K \langle \Delta f, V \rangle_N \,\mathrm{d}v \ . \tag{4}$$

Proof. The first variational formula for the energy is essentially an integration by parts. First write:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} E_K(f_t) &= \frac{1}{2}\int_K \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \operatorname{tr}\langle \mathrm{d}f_t, \mathrm{d}f_t\rangle_N \,\mathrm{d}v_M \\ &= \int_K \operatorname{tr}\langle (\mathrm{d}f, \nabla_{\frac{\partial}{\partial t}} \mathrm{d}f_t)_{|t=0}\rangle_N \,\mathrm{d}v_M\end{aligned}$$

where ∇ here denotes the product connection in $\mathbb{R}\times M$. For this connection, one has $\nabla_{\frac{\partial}{\partial t}} \nabla_u = \nabla_u \nabla_{\frac{\partial}{\partial t}}$ for any $u \in \mathrm{T}M$, from this observation one can derive that

$$(\nabla_{\frac{\partial}{\partial t}} \mathrm{d}f_t)_{|t=0} = \nabla V \ .$$

We therefore get

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} E_K(f_t) = \int_K \operatorname{tr}\langle \mathrm{d}f, \nabla V\rangle_N \,\mathrm{d}v_M \ . \tag{5}$$

By compatibility of ∇ with the metric in N, one can see that the function $\operatorname{tr}\langle \mathrm{d}f, \nabla V\rangle_N$ is equal to $-\langle \operatorname{tr} \nabla^2 f, V\rangle_N$ plus a co-exact function (namely $\mathrm{d}^*\langle \mathrm{d}f(\cdot), V\rangle$), and is zero outside of K. Stokes' theorem thus yields:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} E_K(f_t) &= -\int_K \langle \operatorname{tr} \nabla^2 f, V\rangle_N \,\mathrm{d}v_M \\ &= -\int_K \langle \Delta f, V\rangle_N \,\mathrm{d}v_M\end{aligned}$$

□

Remark 1.6. The end of the proof, after (5), can be rewritten more convincingly using the tools developed in § 1.2:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} E_K(f_t) &= \int_K \operatorname{tr}\langle \mathrm{d}f, \nabla V\rangle_N \,\mathrm{d}v_M \\ &= \langle \mathrm{d}f, \nabla V\rangle_{\mathrm{L}^2} \\ &= \langle \mathrm{d}_\nabla^* \mathrm{d}f, V\rangle_{\mathrm{L}^2} \\ &= -\langle \Delta f, V\rangle_{\mathrm{L}^2} \ .\end{aligned}$$

Remark 1.7. The first variational formula for the energy (4) says precisely that

$$\operatorname{grad} E(f) = -\Delta f$$

on $C := C^\infty(M, N)$, where the gradient is taken with respect to the Riemannian metric on C defined by $\langle U, V\rangle = \int_M \langle U, V\rangle_N \,\mathrm{d}v_M$ for all $U, V \in \mathrm{T}_f C = \Gamma(f^* \mathrm{T}N)$. One must be careful because C is an infinite-dimensional manifold, but one can equip it with a smooth structure making the previous statement precise. We refer to [KM97, Chap. 42] for details on the smooth structure of C.

One says that f is a critical point of the energy functional if $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} E_K(f_t) = 0$ for any compact $K \subseteq M$ and for any variation (f_t) of f supported in K. According to the previous remark, this can be simply put: $\operatorname{grad} E(f) = 0$. In any case, the next corollary immediately from Theorem 1.5:

Corollary 1.8. *Let $f\colon M \to N$ be a smooth map between Riemannian manifolds.*

$$f \text{ is harmonic} \quad \overset{def}{\Leftrightarrow} \quad f \text{ is a critical point of the energy functional} \quad \Leftrightarrow \quad \Delta f = 0\,.$$

1.2 f is harmonic if and only if $\mathrm{d}_\nabla^* \mathrm{d} f = 0$

We have seen in § 1.1 that $\mathrm{d}f$ is a section of $\mathrm{T}^*M \otimes E$ where $E = f^*\mathrm{T}N$. We then defined a connection $\bar\nabla$ in $\mathrm{T}^*M \otimes E$ and defined the Hessian of f as $\bar\nabla(\mathrm{d}f)$. Alternatively, one could see $\mathrm{d}f$ as a 1-form with values in E. More generally, we denote $\Omega^k(M,E)$ the space of smooth k-forms with values in E:

$$\Omega^k(M,E) \coloneqq \Gamma(\Lambda^k \mathrm{T}^*M \otimes E)\,.$$

The connection ∇ in E extends uniquely to a linear map

$$\mathrm{d}_\nabla\colon \Omega^k(M,E) \to \Omega^{k+1}(M,E)$$

called the *exterior covariant derivative* such that

$$\mathrm{d}_\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla s \tag{6}$$

for all $\omega \in \Omega^k(M,\mathbb{R})$ and $s \in \Gamma(M,E)$. Note that d_∇ does not see the metric on M: it only depends on $\nabla = f^*\nabla^N$. In particular, one can consider the 2-form $d_\nabla(\mathrm{d}f) \in \Omega^2(M,E)$, which only depends on the metric on N, but we shall soon see that $d_\nabla(\mathrm{d}f) = 0$.

On the other hand, there is a tensor product connection $\bar\nabla\colon \Gamma(\mathrm{T}^*M \otimes E) \to \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M \otimes E)$ and more generally a tensor product connection

$$\bar\nabla\colon \Gamma((T^*M)^{\otimes k} \otimes E) \to \Gamma((T^*M)^{\otimes(k+1)} \otimes E)\,.$$

Since $\Lambda^k \mathrm{T}^*M$ is a subspace of $(T^*M)^{\otimes k}$ (the subspace of antisymmetric tensors), one can restrict $\bar\nabla$ to this subspace and get a map $\bar\nabla\colon \Omega^k(M,E) \to \Gamma((T^*M)^{\otimes(k+1)} \otimes E)$. It turns out that d_∇ is the antisymmetrization of $\bar\nabla$:

Proposition 1.9. $\mathrm{d}_\nabla\colon \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ *is the antisymmetrization of the restriction of $\bar\nabla$ to $\Omega^k(M,E)$. Concretely, given $\alpha \in \Omega^k(M,E)$:*

$$\mathrm{d}_\nabla\alpha(u_0,\dots,u_k) = \sum_{s=0}^{k} (-1)^s\, (\bar\nabla_{u_s}\alpha)(u_0,\dots,\widehat{u_s},\dots,u_k) \tag{7}$$

where the notation $\widehat{u_s}$ means that u_s is omitted. For example ($k = 1$):

$$(\mathrm{d}_\nabla\alpha)(u,v) = (\bar\nabla\alpha)(u,v) - (\bar\nabla\alpha)(v,u) \tag{8}$$

Proof. It suffices that the antisymmetrization of $\bar\nabla$ verifies the characterization of the exterior covariant derivative given in (6), which is readily done. □

Recall that $\mathrm{d}f \in \Omega^1(M,E)$, so that $\mathrm{d}_\nabla(\mathrm{d}f) \in \Omega^2(M,E)$.

Proposition 1.10. *For any smooth $f\colon M \to N$, $\mathrm{d}_\nabla(\mathrm{d}f) = 0$.*

Proof. By Proposition 1.9, $\mathrm{d}_\nabla(\mathrm{d}f)$ is the antisymmetrization of $\bar\nabla(\mathrm{d}f)$. But recall that the Hessian $\nabla^2 f := \bar\nabla(\mathrm{d}f)$ is symmetric: Proposition 1.2. □

Proposition 1.10 says that $\mathrm{d}f$ is always a closed 1-form. Let us show that f is harmonic if and only if $\mathrm{d}f$ is co-closed. First we need to introduce the Hodge star and the codifferential.

Definition 1.11. Let M be a Riemannian manifold and let $E \to M$ be a smooth vector bundle with a metric $\langle \cdot, \cdot \rangle_E$.

- The *mixed product* of E-valued differential forms is the operation:

$$\begin{aligned}\Omega^k(M,E) \times \Omega^l(M,E) &\to \Omega^{k+l}(M,\mathbb{R})\\ (\alpha,\beta) &\mapsto \langle \alpha \wedge \beta \rangle\end{aligned}$$

  defined by $\langle \omega_1 \otimes s_1 \wedge \omega_2 \otimes s_2 \rangle = \omega_1 \wedge \omega_2 \, \langle s_1, s_2 \rangle_E$.
- The *pointwise inner product* on $\Omega^k(M,E)$ is the operation:

$$\begin{aligned}\Omega^k(M,E) \times \Omega^k(M,E) &\to C^\infty(M,\mathbb{R})\\ (\alpha,\beta) &\mapsto \langle \alpha,\beta \rangle\end{aligned}$$

  defined by $\langle \omega_1 \otimes s_1 \wedge \omega_2 \otimes s_2 \rangle = \langle \omega_1, \omega_2 \rangle_M \, \langle s_1, s_2 \rangle_E$, where $\langle \cdot, \cdot \rangle_M$ is the inner product in $\Lambda^k \mathrm{T}^*_x M$ induced from the inner product $\langle \cdot, \cdot \rangle_M$ in $\mathrm{T}_x M$.
- The *inner product* in $\Omega^k(M,E)$ is the operation:

$$\begin{aligned}\Omega^k(M,E) \times \Omega^k(M,E) &\to \mathbb{R}\\ (\alpha,\beta) &\mapsto \langle \alpha,\beta \rangle_{\mathrm{L}^2} := \int_M \langle \alpha,\beta \rangle \, \mathrm{d}v_M \ .\end{aligned}$$

  For this definition we assume that M is compact or α and β both have compact support.
- The *Hodge star* in $\Omega^\bullet(M,E)$ is the operation:

$$\begin{aligned}\Omega^k(M,E) &\to \Omega^{m-k}(M,E) \quad (m = \dim M)\\ \beta &\mapsto *\beta\end{aligned}$$

  characterized by the identity:

$$\langle \alpha \wedge *\beta \rangle = \langle \alpha, \beta \rangle \, \mathrm{vol}_M \ .$$

The following proposition is elementary and its proof is left to the reader:

Proposition 1.12. *The Hodge star $*$ in $\Omega^\bullet(M,E)$ (cf Definition 1.11) is well-defined. Moreover:*

(i) For any $\alpha = \omega \otimes s \in \Omega^k(M,E)$, $*\alpha = (*\omega) \otimes s$, where $*\omega$ is the standard Hodge star for real-valued differential forms (i.e. take Definition 1.11 with $E = \mathbb{R}$).

(ii) The Hodge star is a pointwise linear isometry:

$$\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle$$

(iii) The Hodge star is an involution up to sign: for all $\alpha \in \Omega^k(M,R)$,

$$* * \alpha = (-1)^{k(m-k)}\alpha$$

We are now ready to define the codifferential and Hodge Laplacian:

Definition 1.13. Let M be a Riemannian manifold, let $E \to M$ be a vector bundle with a metric $\langle\cdot,\cdot\rangle_E$ and with a connection ∇ preserving the metric.

- The *codifferential* in $\Omega^\bullet(M,E)$ is the operation:

$$\begin{aligned} \mathrm{d}_\nabla^*\colon \Omega^k(M,E) &\to \Omega^{k-1}(M,E) \\ \alpha &\mapsto \mathrm{d}_\nabla^*\alpha := (-1)^{m(k-1)+1} * \mathrm{d}_\nabla * \alpha \end{aligned}$$

- The *Hodge Laplacian* in $\Omega^k(M,E)$ is the operation:

$$\begin{aligned} \Delta\colon \Omega^k(M,E) &\to \Omega^k(M,E) \\ \alpha &\mapsto \Delta\alpha := \mathrm{d}_\nabla^*\,\mathrm{d}_\nabla\alpha + \mathrm{d}_\nabla\,\mathrm{d}_\nabla^*\alpha \end{aligned}$$

A k-form $\alpha \in \Omega^k(M,E)$ is called *harmonic* if $\Delta\alpha = 0$.

The following proposition is both elementary and crucial:

Proposition 1.14. *Let M be a Riemannian manifold, let $E \to M$ be a vector bundle with a metric $\langle\cdot,\cdot\rangle_E$ and with a connection ∇ preserving the metric.*

(i) The codifferential d_∇^* is the formal adjoint of the differential d_∇:

$$\langle \mathrm{d}_\nabla\alpha, \beta\rangle_{\mathrm{L}^2} = \langle \alpha, \mathrm{d}_\nabla^*\beta\rangle_{\mathrm{L}^2}$$

whenever this is well-defined ($\deg\beta = \deg\alpha + 1$ *and α or β has compact support).*

(ii) A differential form α with compact support is harmonic if and only if it is closed and co-closed:

$$\Delta\alpha = 0 \quad \Leftrightarrow \quad \mathrm{d}_\nabla\alpha = 0 \;\; \textit{and} \;\; \mathrm{d}_\nabla^*\alpha = 0\,.$$

Proof. For (i) we write, given $\alpha \in \Omega^k(M,E)$ and $\beta \in \Omega^{k+1}(M,E)$, both with compact support:

$$\begin{aligned} \langle \alpha, \mathrm{d}_\nabla^*\beta\rangle \,\mathrm{vol}_M &= \big\langle \alpha \wedge *(\mathrm{d}_\nabla^*\beta)\big\rangle \\ &= \big\langle \alpha \wedge * \big[(-1)^{mk+1} * \mathrm{d}_\nabla * \beta\big]\big\rangle \\ &= \Big\langle \alpha \wedge (-1)^{mk+1}(-1)^{(m-k)k}\,\mathrm{d}_\nabla * \beta\Big\rangle \\ &= (-1)^{k+1}\,\langle \alpha \wedge \mathrm{d}_\nabla * \beta\rangle \end{aligned}$$

Now write

$$\begin{aligned} \mathrm{d}\langle \alpha \wedge *\beta\rangle &= \langle \mathrm{d}_\nabla\alpha \wedge *\beta\rangle + (-1)^k \langle \alpha \wedge \mathrm{d}_\nabla * \beta\rangle \\ &= \langle \mathrm{d}_\nabla\alpha, \beta\rangle\,\mathrm{vol}_M - \langle \alpha, \mathrm{d}_\nabla^*\beta\rangle\,\mathrm{vol}_M \end{aligned}$$

and integrate over M (use Stokes' theorem) to find (i). Now (ii) follows easily:

$$\begin{aligned} \langle \Delta\alpha, \alpha\rangle_{\mathrm{L}^2} &= \langle \mathrm{d}_\nabla^*\,\mathrm{d}_\nabla\alpha, \alpha\rangle_{\mathrm{L}^2} + \langle \mathrm{d}_\nabla\,\mathrm{d}_\nabla^*\alpha, \alpha\rangle_{\mathrm{L}^2} \\ &= \langle \mathrm{d}_\nabla\alpha, \mathrm{d}_\nabla\alpha\rangle_{\mathrm{L}^2} + \langle \mathrm{d}_\nabla^*\alpha, \mathrm{d}_\nabla^*\alpha\rangle_{\mathrm{L}^2} \end{aligned}$$

and observe that $\langle \mathrm{d}_\nabla\alpha, \mathrm{d}_\nabla\alpha\rangle \geqslant 0$ with equality if and only if $\mathrm{d}_\nabla\alpha = 0$, same for $\langle \mathrm{d}_\nabla^*\alpha, \mathrm{d}_\nabla^*\alpha\rangle$. □

Out of interest for the reader, let us mention the main theorem of Hodge theory in the classical case $N = \mathbb{R}$:

Theorem 1.15 (Hodge decomposition)**.** *Let M be a closed manifold. Then*

$$\Omega^k(M,\mathbb{R}) = \mathcal{H}^k(M,\mathbb{R}) \oplus \mathrm{Im}(\mathrm{d}) \oplus \mathrm{Im}(\mathrm{d}^*)$$

where $\mathcal{H}^k(M,\mathbb{R}) = \ker \Delta$ is the space of harmonic k-forms.

It follows from the Hodge decomposition theorem that the de Rham cohomology space $H^k_{\mathrm{dR}}(M,\mathbb{R})$ is isomorphic to $\mathcal{H}^k(M,\mathbb{R})$. In particular it is finite-dimensional, as the kernel of an elliptic operator on a closed manifold.

Now let us come back to the setting where $E = f^*\mathrm{T}N$ and $\nabla = f^*\nabla^N$. We have seen that $\mathrm{d}f \in \Omega^1(M,E)$ is always a closed 1-form (Proposition 1.10). Weighing in Proposition 1.14 (ii), $\mathrm{d}f$ is a harmonic 1-form if and only if it is co-closed:

Proposition 1.16. *If M is compact,*

$$\mathrm{d}f \in \Omega^1(M,E) \text{ is harmonic} \quad \Leftrightarrow \quad \mathrm{d}^*_\nabla \mathrm{d}f = 0$$

Note that $\mathrm{d}^*_\nabla \mathrm{d}f$ is an element of $\Omega^0(E)$, i.e. a section of $f^*(\mathrm{T}N)$, just like the Laplacian Δf. This last proposition closes the loop:

Proposition 1.17. *For any smooth $f\colon M \to N$,*

$$\mathrm{d}^*_\nabla \mathrm{d}f = -\Delta f$$

Proof. This is a special case of the formula

$$\mathrm{d}^*_\nabla \alpha = -\operatorname{tr}_{12} \bar{\nabla}\alpha \tag{9}$$

for any $\alpha \in \Omega^k(M,E)$, which is the d^*_∇-analog of (7), more concretely:

$$\mathrm{d}^*_\nabla \alpha(u_1,\dots,u_{k-1}) = -\sum_{j=1}^m \bar{\nabla}_{e_j}\alpha(e_j,u_1,\dots,u_{k-1}) \tag{10}$$

where $(e_j)_{1\leqslant j\leqslant m}$ is any local orthonormal frame field on M. One can prove this formula by verifying that using it as a definition for $\mathrm{d}^*_\nabla\alpha$, it does give a formal adjoint of d_∇: check that $\langle \mathrm{d}^*_\nabla\alpha, \beta\rangle$ is pointwise equal to $\langle \alpha, \mathrm{d}_\nabla\beta\rangle$ plus a globally defined co-exact function. (Alternatively, a direct proof can be given using normal coordinates: cf [EL83, Lemma 1.20].) □

Remark 1.18. Equation (9) (or (10)) says that $\mathrm{d}^*_\nabla = -\operatorname{div}$, where div is the divergence operator suitably interpreted on $\Omega^k(M,E)$.

Remark 1.19. Proposition 1.17 holds even if M is not compact: the proof above shows that $\langle \mathrm{d}^*_\nabla\alpha + \operatorname{tr}_{12}\bar{\nabla}\alpha, \beta\rangle_{\mathrm{L}^2} = 0$ for any *compactly supported* β, but this is enough to conclude.

Remark 1.20. Warning! There is a sign discrepancy between the Laplacian on maps (as in Definition 1.3) and the Hodge Laplacian on forms (as in Definition 1.13). Indeed, when $N = \mathbb{R}$, both Laplacians make sense on $\Omega^0(M,\mathbb{R})$, and differ by a minus sign. This is the well-known disagreement between the "analyst's Laplacian" and the "geometer's Laplacian".

We are now ready to wrap up with a theorem:

Theorem 1.21. *Let $f\colon M \to N$ be a smooth map between Riemannian manifolds.*

$$f \text{ harmonic} \quad \Leftrightarrow \quad \Delta f = 0 \quad \Leftrightarrow \quad \mathrm{d}^*_\nabla \mathrm{d}f = 0 \quad \Leftrightarrow \quad \mathrm{d}f \text{ is a harmonic 1-form.}$$

Remark 1.22. M is assumed compact for the last equivalence. Otherwise it is not necessarily the case that $\Delta f = 0$ if $\Delta\,\mathrm{d}f = 0$: take for example $M = \mathbb{R}^m$, $N = \mathbb{R}$, and $f(x_1,\dots,x_m) = x_1^2$.

2 The Bochner technique

This section is not essential for our exposition and the reader in a hurry may skip it. We explain the Bochner and Weitzenböck formulas and how it is typically used to produce rigidity results. The theorem of Siu and Sampson (cf § 4) will be a variation of this technique when the domain manifold is Kähler.

2.1 Bochner formula

The classical Bochner formula for a smooth function $f\colon M \to \mathbb{R}$ is

$$\frac{1}{2}\Delta\|\nabla f\|^2 - \langle \nabla f, \nabla \Delta f\rangle = \|\nabla^2 f\|^2 + \mathrm{Ric}^M(\nabla f, \nabla f)\,.$$

In this formula, ∇f is an alias for either $\mathrm{d} f$ or $\mathrm{grad}\, f$ (one can harmlessly switch from one to the other using the metric duality). Eells-Sampson proved a generalized version for a smooth function $f\colon M \to N$ between Riemannian manifolds:

$$\frac{1}{2}\Delta\|\nabla f\|^2 - \langle \nabla f, \nabla \Delta f\rangle = \|\nabla^2 f\|^2 + \mathrm{Ric}^M(\langle \nabla f, \nabla f\rangle_N) - R^N(\langle \nabla f \wedge \nabla f\rangle_M, \langle \nabla f \wedge \nabla f\rangle_M) \quad (11)$$

Let us clarify some notations:

- $\langle \nabla f, \nabla \Delta f\rangle$ is the pointwise inner product in $\Omega^1(M, E)$.
- $\langle \nabla f, \nabla f\rangle_N$ is the section of $\mathrm{T}M \otimes \mathrm{T}M$ obtained from pairing ∇f to itself using $\langle\cdot,\cdot\rangle_N$.
- $\langle \nabla f \wedge \nabla f\rangle_M$ is the section of $\Lambda^2\, \mathrm{T}N$ obtained from wedging ∇f to itself and using $\langle\cdot,\cdot\rangle_M$.
- R^N is the curvature operator in N: see § 2.3.

We shall prove the Bochner formula in § 2.4 via the Weitzenböck formula.

2.2 Application of the Bochner formula

Before clarifying curvature tensors (§ 2.3) and proving the Weitzenböck and Bochner formulas (§ 2.4), let us explain the importance of the Bochner formula for the study of harmonic maps.

2.2.1 Rigidity

The key idea is to require the appropriate curvature assumptions so that one controls the sign of all terms on the right-hand side of the Bochner formula (11). Ideally: M has nonnegative Ricci curvature and N has nonpositive sectional curvature. In that case, all the terms on the right-hand side of (11) are pointwise nonnegative. The other key idea for obtaining rigidity results is simply to integrate (11) over M when M is compact and f is harmonic. Indeed, when f is harmonic $\Delta f = 0$, also $\Delta\|\nabla f\|^2$ always integrates to zero. Let us perhaps recall why:

Lemma 2.1. *Let M be a compact Riemannian manifold. For any $u \in C^\infty(M, \mathbb{R})$, $\int_M \Delta u \,\mathrm{d}v_M = 0$.*

Proof. Recall that a top form ω on M has zero integral if and only if it is exact. Using the definition of the codifferential d^* and recalling that $*1 = \mathrm{vol}_M$, this amounts to saying that $u\,\mathrm{vol}_M$ has zero integral if and only if u is co-exact. But $\Delta u = \mathrm{d}^*\mathrm{d}u$ is obviously co-exact. (Equivalently, $\int_M \Delta u\,\mathrm{d}v_M = 0$ is a direct consequence of the divergence theorem, recalling that $\Delta u = \mathrm{div}(\mathrm{grad}\, u)$.) □

Let us summarize the previous observations:

$$\underbrace{\frac{1}{2}\Delta\|\nabla f\|^2}_{\text{integral}=0} - \underbrace{\langle \nabla f, \nabla\Delta f\rangle}_{0 \text{ if } f \text{ harmonic}} = \underbrace{\|\nabla^2 f\|^2}_{\geqslant 0 \text{ pointwise}} + \underbrace{\mathrm{Ric}^M(\langle \nabla f, \nabla f\rangle_N)}_{\geqslant 0 \text{ pointwise}} \underbrace{-R^N(\langle \nabla f \wedge \nabla f\rangle_M, \langle \nabla f \wedge \nabla f\rangle_M)}_{\geqslant 0 \text{ pointwise}}$$

Integrating over M, one sees that under the previous curvature assumptions, if f is harmonic then each of the three terms on the right-hand side must be identically zero. In particular $\nabla^2 f = 0$ everywhere: f is totally geodesic. Furthermore, the vanishing of the curvature terms easily yields:

Theorem 2.2 (Eells-Sampson's strong rigidity theorem)**.** *Let $f\colon M \to N$ be a smooth harmonic map between Riemannian manifolds. Assume M is compact and has nonnegative Ricci curvature and N has nonpositive sectional curvature. Then f is totally geodesic. Moreover:*

(i) If Ric^M is not identically zero, then f is constant.

(ii) If N has negative sectional curvature, then f is constant or maps to a closed geodesic.

2.2.2 Heat flow

The Bochner formula is also crucial for the heat flow technique developed by Eells-Sampson [ES64] and successfully to adapted to many settings by various authors. Let us quickly explain this, although we will not use the heat flow in this report. Assume $(f_t)_{t\in I}$ is a 1-parameter family of maps $M \to N$ satisfying the heat flow equation:

$$\partial_t f_t = \Delta f_t \tag{12}$$

One can show local existence of this flow given f_0 using classical nonlinear parabolic PDE techniques (linearization of the operator $\partial_t - \Delta$ and implicit function theorem, see e.g. [Jos84]). Note that the heat flow is just the gradient flow for the energy functional, since $\operatorname{grad} E(f) = -\Delta(f)$ (Remark 1.7). Assume N is nonpositively curved. The second variational formula for the energy shows that it is a convex functional (see e.g. [GLM18, Prop. 3.4]), making it reasonable to expect that the heat flow might converge to an energy minimizer. When M and N are both compact, one can hope proving convergence of the flow (possibly up to subsequence) using some compactness argument such as the Arzelà–Ascoli theorem. However there are significant obstacles to overcome, such as proving the long-time existence of the heat flow and equicontinuity of (f_t). The Bochner formula shows that $\|\nabla f_t(x)\|$ is uniformly bounded in time and space, solving both these obstacles.

Let us give some details. Denote by $e(f_t) := \frac{1}{2}\|\mathrm{d} f_t\|^2$ the energy density of f_t. A key observation is that if $(f_t)_{t\in I}$ satisfies the heat flow equation (12), then the term $\langle \nabla f_t, \nabla\Delta f_t\rangle$ in the Bochner formula becomes:

$$\langle \nabla f_t, \nabla\Delta f_t\rangle = \partial_t e(f_t)\ .$$

Assuming Ric^M is bounded below by $K \in \mathbb{R}$ (e.g. M is compact) and N has nonpositive sectional curvature, the Bochner formula (11) yields:

$$(\partial_t - \Delta)e(f_t) \leqslant K' e(f_t)$$

where $K' = -2K$. By a generalized mean value property (more precisely: Moser's Harnack inequality for subsolutions of the heat equation [Mos64][LW08, Lemma 5.3.4]), this implies

$$\|e(f_t)\|_\infty \leqslant C E(f_0)$$

for some constant $C > 0$. In other words, the family (f_t) has a uniformly bounded gradient. It obviously implies that it is equicontinuous, but also long-time existence of the heat flow (when N is compact) by a standard "blow up in finite time" argument for nonlinear parabolic PDEs. Thus one can extract $t_k \to +\infty$ such that f_{t_k} converges uniformly to some map f_∞. There still remains work to do, involving regularity theory and Sobolev spaces, to show that f_∞ is a smooth harmonic map, refer to e.g. [Jos84] for details. Let us record the following conclusion:

Theorem 2.3 (Eells-Sampson [ES64])**.** *Let $f\colon M \to N$ be a smooth map, where M and N are compact Riemannian manifolds and N has nonpositive sectional curvature. Then f is homotopic to a harmonic map.*

2.3 Curvature tensors

Now is a good time to clarify our notations and conventions for curvature tensors.

Given a vector bundle $E \to M$ with a connection ∇, it is easy to check that the linear map $\mathrm{d}_\nabla^2 \colon \Omega^0(M,E) \to \Omega^2(M,E)$ is tensorial, i.e. $C^\infty(M,\mathbb{R})$-linear. Therefore there exists a tensor field $F^\nabla \in \Omega^2(\operatorname{End} E)$ such that $\mathrm{d}_\nabla^2(s) = F^\nabla s$. The operator F^∇ is called the *curvature* of ∇. Recalling that d_∇ is the antisymmetrization of ∇ as in (8), F^∇ is concretely given by

$$\begin{aligned} F^\nabla(X,Y)s &= \nabla^2_{X,Y}s - \nabla^2_{Y,X}s \\ &= \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s\,. \end{aligned} \tag{13}$$

where ∇ is assumed torsion-free for the second equality, which says: $F^\nabla(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$.

2.3.1 Riemannian curvature tensors

If M is a Riemannian manifold, the *Riemann curvature tensor* $R := F^\nabla \in \Gamma(\Lambda^2\,\mathrm{T}^*M \otimes \operatorname{End}(\mathrm{T}M))$ is the curvature of the Levi-Civita connection ∇:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,.$$

The *purely covariant version* of the Riemann curvature tensor is the 4-covariant tensor field:

$$R(X,Y,Z,W) = -\langle R(X,Y)Z, W\rangle \tag{14}$$

and the *curvature operator* is the symmetric bilinear form on $\otimes^2\,\mathrm{T}M$ defined by:

$$Q(X\otimes Y, Z\otimes W) = R(X,Y,Z,W)$$

for decomposable tensors, and extended bilinearly to $\otimes^2\,\mathrm{T}M$ (it can also be defined as a bilinear form on $\Lambda^2\,\mathrm{T}M$ or as an endomorphism of $\otimes^2\,\mathrm{T}M$ or $\Lambda^2\,\mathrm{T}M$). The *sectional curvature* K is

$$K(X,Y) = \frac{Q(X\otimes Y, X\otimes Y)}{\|X\wedge Y\|^2}$$

with $\|X\wedge Y\|^2 = \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2$. It is defined for any two linearly independent vectors X and Y with same basepoint and only depends on the plane spanned by X and Y. Finally, the *Ricci* *curvature tensor* is the symmetric bilinear form on $\mathrm{T}M$ defined by

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \operatorname{tr} R(X,\cdot,Y,\cdot)\\ &= \sum_{j=1}^{n} R(X,e_j,Y,e_j)\end{aligned}$$

for any local orthonormal frame field $(e_j)_{1\leqslant j\leqslant n}$.

Remark 2.4. It is certainly questionable to define the purely covariant Riemann tensor with a minus sign as in (14) and still use the same letter R! The main reason for this choice is nicer formulas for sectional curvatures. Some authors introduce the minus sign earlier (e.g. $R = -d_\nabla^2$), others later (e.g. $K(X,Y) = -\frac{R(X,Y,X,Y)}{\|X\wedge Y\|^2}$), but no solution is entirely satisfactory. The only unanimous convention is sectional curvature: it should agree with the Gaussian curvature when $\dim M = 2$.

2.3.2 Curvature tensors in $\Omega^k(M,E)$

In order to express the Weitzenböck formula below, let us give a generalization of Riemann and Ricci curvature to bundles of vector-valued forms. Let M be a Riemannian manifold, let $E \to M$ be a vector bundle with a metric $\langle\cdot,\cdot\rangle_E$ and with a connection ∇ preserving the metric. Recall that there is a product connection $\bar\nabla$ in the bundle $\Lambda^k\,\mathrm{T}^*M \otimes \mathrm{T}N$. By definition, the curvature of that bundle is the curvature of $\bar\nabla$ (see (13)). Explicitly:

$$(R(X,Y)\alpha)(u_1,\dots,u_k) = F^\nabla(X,Y)(\alpha(u_1,\dots,u_k)) - \sum_{j=1}^{k}\alpha(u_1,\dots,R^M(X,Y)u_j,\dots,u_k)\,.$$

We then define the *Ricci (or Bochner, or Weitzenböck) operator* $S\colon \Omega^k(M,E)\to\Omega^k(M,E)$:

$$S(\alpha)(u_1,\dots,u_k) = \sum_{s=1}^{k}(-1)^{s-1}\operatorname{tr}\left[(R(\,\cdot\,,u_s)\sigma)(\,\cdot\,,u_1,\dots,\widehat{u_s},\dots,u_k)\right]$$

Note that when E is the trivial flat bundle $M\times\mathbb{R}\to M$ and $k=1$, $S(\alpha) = \mathrm{Ric}^M(\alpha)$.

2.4 Weitzenböck and Bochner formulas

We conclude with the proof of the Bochner formula via the Weitzenböck formula, a neat argument.

Let M be a Riemannian manifold, let $E\to M$ be a vector bundle with a metric $\langle\cdot,\cdot\rangle_E$ and with a connection ∇ preserving the metric. In $\Omega^k(M,E)$ we have two Laplacian operators:

- The Hodge Laplacian Δ introduced in Definition 1.13.
- The *connection Laplacian* (or *trace Laplacian* or *rough Laplacian*) $\operatorname{tr}\left(\bar\nabla^2\right)$, which can be seens as the natural extension of the Laplacian on functions (see Definition 1.3).

Remark 2.5. The connection Laplacian $\operatorname{tr}(\nabla^2)$ is well-defined in $\Gamma(E)$ for any vector bundle E with a metric and a compatible connection ∇ (above we just specialized to the bundle $\Lambda^k\,\mathrm{T}^*M\otimes E$). Some call *Bochner Laplacian* the operator $\nabla^*\nabla$ where $\nabla^*\colon \Gamma(\mathrm{T}^*M\otimes E)\to\Gamma(E)$ is the formal adjoint of ∇. Since $\nabla^* = -\operatorname{tr}\nabla$, the Bochner Laplacian is simply minus the connection Laplacian.

The Weitzenböck formula relates both Laplacian operators:

Theorem 2.6 (Weitzenböck formula)**.** *For all $\alpha \in \Omega^k(M, E)$,*

$$\Delta\alpha = -\operatorname{tr}(\bar{\nabla}^2\alpha) + S(\alpha) \tag{15}$$

Note that for $k = 0$, the Weitzenböck formula reduces to Proposition 1.17, since $S = 0$. *Be wary that the usual Laplacian (i.e. connection Laplacian) and the Hodge Laplacian differ by a minus sign on functions!* For $k \geqslant 1$, it is straightforward to prove the Weitzenböck formula by direct computation, we leave it to the reader. It is written e.g. in [Xin96, Prop. 1.3.4]); we also refer to [EL83, (1.29)] for this formula and more historical references. The reader eager to learn about the full extent of Weitzenböck formulas may refer to [Nic13] as a starting point.

Corollary 2.7 (Bochner formula in $\Omega^k(M, E)$)**.** *For all $\alpha \in \Omega^k(M, E)$,*

$$\frac{1}{2}\Delta\|\alpha\|^2 + \langle\Delta\alpha, \alpha\rangle = \|\nabla\alpha\|^2 + \langle S(\alpha), \alpha\rangle \tag{16}$$

Proof.

$$\begin{aligned}\frac{1}{2}\Delta\|\alpha\|^2 &= \frac{1}{2}\operatorname{tr}\nabla^2\langle\alpha, \alpha\rangle \\ &= \operatorname{tr}\nabla\langle\nabla\alpha, \alpha\rangle \\ &= \operatorname{tr}\langle\nabla^2\alpha, \alpha\rangle + \operatorname{tr}\langle\nabla\alpha, \nabla\alpha\rangle\end{aligned}$$

For the first term, we use the Weitzenböck formula (15): $\operatorname{tr}(\bar{\nabla}^2\alpha) = S(\alpha) - \Delta\alpha$. For the second term, we observe that $\operatorname{tr}\langle\nabla\alpha, \nabla\alpha\rangle$ is just $\langle\nabla\alpha, \nabla\alpha\rangle$ by definition of the pointwise inner product in $\Omega^{k+1}(M, E)$. We thus get:

$$\frac{1}{2}\Delta\|\alpha\|^2 = \langle S(\alpha) - \Delta\alpha, \alpha\rangle + \|\nabla\alpha\|^2$$

□

Corollary 2.8 (Bochner formula for maps between Riemannian manifolds)**.** *Let $f\colon M \to N$ be a smooth map between Riemannian manifolds.*

$$\frac{1}{2}\Delta\|\nabla f\|^2 - \langle\nabla f, \nabla\Delta f\rangle = \|\nabla^2 f\|^2 + \operatorname{Ric}^M(\langle\nabla f, \nabla f\rangle_N) - R^N(\langle\nabla f \wedge \nabla f\rangle_M, \langle\nabla f \wedge \nabla f\rangle_M)\,. \tag{17}$$

Proof. The Bochner formula (17) is just a specialization of (16) when $E = f^*\,\mathrm{T}N$, $k = 1$ and $\alpha = \mathrm{d}f$. Indeed, one readily checks that $\langle S(\alpha), \alpha\rangle$ gives the two curvature terms of (17). It just remains to argue that $\Delta(\mathrm{d}f) = -\nabla(\Delta f)$. Since $\mathrm{d}f$ is d_∇-closed (Proposition 1.10), $\Delta(\mathrm{d}f) = \mathrm{d}_\nabla\,\mathrm{d}_\nabla^*\,\mathrm{d}f$. Conclude recalling that $\mathrm{d}_\nabla^*\,\mathrm{d}f = -\Delta f$ (Proposition 1.17). □

3 Kähler manifolds and pluriharmonic maps

3.1 Complex and Kähler manifolds

Let M be a complex manifold. Recall that M has an *almost complex structure*, i.e. $J \in \Gamma(\operatorname{End}\mathrm{T}M)$ such that $J^2 = -1$, coming from the multiplication by the scalar i in $\mathbb{C}^n$ (pulled back by complex charts). Such an almost complex structure J on the underlying real manifold M is called *integrable* because it is induced by a complex structure on M. The Newlander-Nirenberg theorem states that J is integrable if and only if its Nijenhuis tensor vanishes, see e.g. [Dem12]. Note that J induces an orientation of M and we always assume this agrees with the given orientation of M.

The complexified tangent bundle of M splits into $\pm i$-eigenspaces of J as $\mathrm{T}M \otimes \mathbb{C} = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M$. Accordingly, a complexified tangent vector decomposes into types as $u = u^{1,0} + u^{0,1}$, with:

$$u^{1,0} = \frac{1}{2}(u - iJu) \qquad u^{0,1} = \frac{1}{2}(u + iJu) \ . \tag{18}$$

The cotangent bundle T^*M also admits a linear complex structure still denoted J, defined by $J\alpha := \alpha \circ J$, and we have a similar decomposition into $\pm i$-eigenspaces: $\mathrm{T}^*M \otimes \mathbb{C} = \mathrm{T}^{*1,0}M \oplus \mathrm{T}^{*0,1}M$. The same formula (18) works for the type decomposition $\alpha = \alpha^{1,0} + \alpha^{0,1}$. Thankfully, it is true that $\alpha^{1,0}(u) = \alpha(u^{1,0})$ and $\alpha^{0,1}(u) = \alpha(u^{0,1})$, where on the right-hand side one takes the complexification (complex linear extension) of α.

In the same vein, if $f\colon M \to N$ is a smooth map where M is a (almost) complex manifold and N is a real manifold, then

$$\mathrm{d}^{1,0}f = \frac{1}{2}(\mathrm{d}f - i\,\mathrm{d}f \circ J) \qquad \mathrm{d}^{0,1}f = \frac{1}{2}(\mathrm{d}f + i\,\mathrm{d}f \circ J) \ .$$

Both are linear maps $\mathrm{T}M \to \mathrm{T}N \otimes \mathbb{C}$, or equivalently, complex linear maps $\mathrm{T}M \otimes \mathbb{C} \to \mathrm{T}N \otimes \mathbb{C}$. In the latter description, $\mathrm{d}^{1,0}f(u)$ is the same as $\mathrm{d}f(u^{1,0})$, using the complex linear extension of $\mathrm{d}f$. Be wary that we do not use any complex structure on N at this point: in particular, when $N = \mathbb{C}$, $\mathrm{d}^{1,0}f$ (resp. $\mathrm{d}^{0,1}f$) is different from what is usually denoted ∂f (resp. $\bar{\partial} f$). We also denote:

$$\mathrm{d}^{\mathrm{c}}f := -\mathrm{d}f \circ J = -i(\mathrm{d}^{1,0}f - \mathrm{d}^{0,1}f) \ .$$

If (M, J_M) and (N, J_N) are both (almost) complex manifolds, then a smooth map $f\colon M \to N$ is called holomorphic if $f^*J_N = J_M$, i.e. $\mathrm{d}f \circ J_M = J_N \circ \mathrm{d}f$. Equivalently, f preserves types: $\mathrm{d}f(\mathrm{T}^{1,0}M) \subseteq \mathrm{T}^{1,0}N$ (resp. $\mathrm{d}f(\mathrm{T}^{0,1}M) \subseteq \mathrm{T}^{0,1}N$).

Let us now recall the definition of Hermitian and Kähler manifolds:

Definition 3.1. A *Hermitian manifold* is a smooth manifold equipped with a Riemannian metric g and an integrable almost complex structure J that are *compatible*, meaning that J is a linear isometry: $g(Ju, Jv) = g(u, v)$.

Given a Hermitian manifold (M, g, J), one defines the *fundamental form* ω by $\omega(u, v) = g(Ju, v)$. It is a nondegenerate 2-form on M, in fact it is a $(1,1)$-form, i.e. $\omega(Ju, Jv) = \omega(u, v)$. One can also define a Hermitian inner product h on $(\mathrm{T}M, J)$ by letting $h = g - i\omega$. For example, when $M = \mathbb{C}$, $h = \mathrm{d}z \otimes \mathrm{d}\bar{z} = g - i\omega$ with $g = \mathrm{d}x^2 + \mathrm{d}y^2$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y$.

The fundamental form ω can be incoroporated in the data of a Hermitian manifold (M, g, J, ω), it being understood that $\omega(u, v) = g(Ju, v)$. This relation shows that any 2 of the 3 tensors g, J, ω determine the third, this matches the well-known 2-out-of-3 property of the unitary group $\mathrm{U}(n)$.

Proposition 3.2. *Let (M, g, I, ω) be a Hermitian manifold.*

(i) Denoting $n := \dim_{\mathbb{C}} M$, one has

$$\frac{\omega^n}{n!} = \mathrm{vol}_M \ .$$

(ii) For all $\alpha \in \Omega^1(M, \mathbb{R})$

$$*\alpha = \frac{\omega^{n-1}}{(n-1)!} \wedge J\alpha \ .$$

(See § 1.2 for the definition of the Hodge star.)

Proof. Both statements are linear algebra statements in $\mathrm{T}_x M$. One easily proves them using an orthonormal basis $(e_1, Je_1, \dots, e_n, Je_n)$ of $\mathrm{T}_x M$. ◻

Definition 3.3. A Hermitian manifold (M, g, J, ω) is called *Kähler* if $\mathrm{d}\omega = 0$.

In other words, ω is required to be a symplectic structure (nondegenerate closed 2-form). It turns out that the simple condition $\mathrm{d}\omega = 0$ has deep consequences on the geometry of M, starting with the existence of holomorphic coordinates identifying (M, g, J, ω) to $\mathbb{C}^n$ with the flat metric to first order. However for this report we shall not need to develop any theory of Kähler manifolds: $\mathrm{d}\omega = 0$ is the only characterization that we will use, especially for the proof of the Siu-Sampson theorem following Toledo in § 4.3.2. In fact, § 4 will be a perfect illustration of the provocative [ABC+96, Methatheorem 1.2]:

Kähler manifolds are complex manifolds whose geometry reduces to linear algebra.

The only tool from analysis we shall use for the proof in § 4.3.2 is the elementary fact that $\int_M \eta = 0$ if η is exact, the rest is linear algebra!

It is not hard to show that the condition $\mathrm{d}\omega = 0$ is equivalent to $\nabla J = 0$. (More generally, $\nabla J = 0$ if and only if J is integrable and ω is closed.) A straightforward yet key consequence is:

Proposition 3.4. *Let (M, g, J, ω) be a Hermitian manifold. If M is Kähler then the Riemann and Ricci tensors of M are of type $(1, 1)$, i.e. $R^M(Ju, Jv) = R^M(u, v)$ and $\mathrm{Ric}^M(Ju, Jv) = \mathrm{Ric}^M(u, v)$.*

3.2 Pluriharmonic maps

In this subsection, N is any Riemannian manifold.

If (M, g, J, ω) is a Kähler manifold, we can nicely combine Proposition 1.17 with Proposition 3.2 (ii) and use closedness of ω to obtain:

Proposition 3.5. *Let $f : M \to N$ be a smooth map, where (M, g, J, ω) is a Kähler manifold of complex dimension n. Then*

$$-\Delta f = * \left(\frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{d}_\nabla \mathrm{d}^c f \right) .$$

Proof.

$$\begin{aligned}
-\Delta f &= \mathrm{d}_\nabla^* \,\mathrm{d} f && \text{by Proposition 1.17} \\
&= (-1)^{2n(k-1)+1} * \mathrm{d}_\nabla * \mathrm{d} f && \text{by definition of } \mathrm{d}_\nabla^* \text{ (cf Definition 1.11)} \\
&= * \,\mathrm{d}_\nabla \left(\frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{d}^c f \right) && \text{by Proposition 3.2 (ii)} \\
&= * \left(\frac{\mathrm{d}(\omega^{n-1})}{(n-1)!} \wedge \mathrm{d}^c f + \frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{d}_\nabla \mathrm{d}^c f \right) && \text{by definition of } \mathrm{d}_\nabla \text{ (see (6))} \\
&= * \left(\frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{d}_\nabla \mathrm{d}^c f \right) && \text{since } \mathrm{d}\omega = 0 \text{ (M Kähler)}
\end{aligned}$$

◻

An important special case is when $\dim_{\mathbb{C}} M = 1$, i.e. $M = C$ is a Riemann surface (complex curve), in which case any compatible metric is Kähler. Proposition 3.5 then yields $-\Delta f = *(\mathrm{d}_\nabla \mathrm{d}^c f)$. Therefore $f\colon C \to N$ is harmonic if and only if $\mathrm{d}_\nabla \mathrm{d}^c f = 0$. Recall that d_∇ does not see the metric on C (only depends on $\nabla = f^*\nabla^N$), and neither does d^c. Therefore $f\colon C \to N$ being harmonic only depends on the complex structure on C:

Corollary 3.6. *Let C be a complex curve. The harmonicity of a smooth map $f\colon C \to N$ does not depend on the choice of a compatible metric on C.*

A consequence of Corollary 3.6 is the sanity of the definition of pluriharmonicity:

Definition 3.7. Let $f\colon M \to N$ be a smooth map where M is a complex manifold. The map f is called *pluriharmonic* if for every 1-dimensional complex submanifold $C \subseteq M$, the restricted map $f_{|C}\colon C \to N$ is harmonic.

Pluriharmonicity can alternatively be defined by the equation $\mathrm{d}_\nabla \mathrm{d}^c f = 0$:

Proposition 3.8. *Let $f\colon M \to N$ be a smooth map where M is a complex manifold.*

$$f \text{ pluriharmonic} \quad \Leftrightarrow \quad \mathrm{d}_\nabla \mathrm{d}^c f = 0\,.$$

Further characterizations:

$$f \text{ pluriharmonic} \quad \Leftrightarrow \quad \mathrm{d}_\nabla^{0,1} \mathrm{d}^{1,0} f = 0 \quad \Leftrightarrow \quad \mathrm{d}_\nabla^{1,0} \mathrm{d}^{0,1} f = 0 \quad \Leftrightarrow \quad \left(\nabla^2 f\right)^{1,1} = 0\,.$$

Remark 3.9. For the last one, choose any compatible Riemannian metric on M so that the Hessian $\nabla^2 f$ makes sense (see Definition 1.3). Due to the characterization $\left(\nabla^2 f\right)^{1,1} = 0$, pluriharmonic maps are sometimes alternatively called *(1,1)-geodesic maps*.

Proof of Proposition 3.8. Recall that $\mathrm{d}_\nabla \mathrm{d} f = 0$ (Proposition 1.10). Decomposing into types:

$$\underbrace{\mathrm{d}_\nabla^{1,0} \mathrm{d}^{1,0} f}_{(\mathrm{d}_\nabla \mathrm{d} f)^{2,0}=0} + \underbrace{\mathrm{d}_\nabla^{1,0} \mathrm{d}^{0,1} f + \mathrm{d}_\nabla^{0,1} \mathrm{d}^{1,0} f}_{(\mathrm{d}_\nabla \mathrm{d} f)^{1,1}=0} + \underbrace{\mathrm{d}_\nabla^{0,1} \mathrm{d}^{0,1} f}_{(\mathrm{d}_\nabla \mathrm{d} f)^{0,2}=0} = 0$$

On the other hand:

$$\begin{aligned} \mathrm{d}_\nabla \mathrm{d}^c f &= -i\, \mathrm{d}_\nabla \left(\mathrm{d}^{1,0} f - \mathrm{d}^{0,1} f\right) \\ &= -i \left(\mathrm{d}_\nabla^{1,0} \mathrm{d}^{1,0} f + \mathrm{d}_\nabla^{1,0} \mathrm{d}^{0,1} f - \mathrm{d}_\nabla^{0,1} \mathrm{d}^{1,0} f - \mathrm{d}_\nabla^{0,1} \mathrm{d}^{0,1} f\right) \end{aligned}$$

We thus find

$$\mathrm{d}_\nabla \mathrm{d}^c f = -2i\, \mathrm{d}_\nabla^{1,0} \mathrm{d}^{0,1} f = 2i\, \mathrm{d}_\nabla^{0,1} \mathrm{d}^{1,0} f\,.$$

It follows of course that $\mathrm{d}_\nabla \mathrm{d}^c f = 0 \Leftrightarrow \mathrm{d}_\nabla^{0,1} \mathrm{d}^{1,0} f = 0 \Leftrightarrow \mathrm{d}_\nabla^{1,0} \mathrm{d}^{0,1} f = 0$. We also learn that $\mathrm{d}_\nabla \mathrm{d}^c f$ is of type $(1,1)$.

If $\mathrm{d}_\nabla \mathrm{d}^c f = 0$, then this equation still holds in restriction to any complex curve $C \subseteq M$. We saw that on a complex curve, the equation $\mathrm{d}_\nabla \mathrm{d}^c f = 0$ is equivalent to f being harmonic. Thus f is pluriharmonic. Conversely, if f is pluriharmonic, then the equation $\mathrm{d}_\nabla \mathrm{d}^c f = 0$ must hold in restriction to any complex curve. As a 2-form of type $(1,1)$, $\mathrm{d}_\nabla \mathrm{d}^c f$ vanishes identically if and only if it vanishes on pairs of vectors of the form (u, Ju). Since any such pair is tangent to some complex curve $C \subseteq M$, the conclusion follows: $\mathrm{d}_\nabla \mathrm{d}^c f = 0$.

It remains to show that $\mathrm{d}_\nabla \mathrm{d}^c f = 0 \Leftrightarrow \left(\nabla^2 f\right)^{1,1} = 0$. By Proposition 1.9,

$$\begin{aligned}(\mathrm{d}_\nabla \mathrm{d}^c f)(u, v) &= \bar{\nabla}_u(\mathrm{d}^c f(v)) - \bar{\nabla}_v(\mathrm{d}^c f(u)) \\ &= -\nabla^2 f(u, Jv) + \nabla^2 f(v, Ju)\,.\end{aligned}$$

It follows that

$$\begin{aligned}-(\mathrm{d}_\nabla \mathrm{d}^c f)(Ju, v) &= \nabla^2 f(Ju, Jv) + \nabla^2 f(u, v) \\ &= 2\left(\nabla^2 f\right)^{1,1}(u, v)\end{aligned}$$

($2T^{1,1}(u, v) = T(u, v) + T(Ju, Jv)$ holds for any 2-covariant tensor T.) The conclusion follows. □

Note that as it turns out, the tensor $\left(\nabla^2 f\right)^{1,1} = -\frac{1}{2}(\mathrm{d}_\nabla \mathrm{d}^c f)(Ju, v)$ does not depend on the choice of a compatible metric on M. One may call it the *Levi form* of f.

Here are some properties of pluriharmonic maps:

Proposition 3.10. *Let $f : M \to N$ where M is a complex manifold and N is a Riemannian manifold.*
(i) If f is pluriharmonic, the restriction of f to any complex submanifold of M is pluriharmonic.
(ii) If f is pluriharmonic, then f is harmonic with respect to any compatible Kähler metric on M.
(iii) If N is a Kähler manifold and f is holomorphic or antiholomorphic, then f is pluriharmonic.

Remark 3.11. When $N = \mathbb{R}$, a function $f : M \to \mathbb{R}$ is pluriharmonic if and only if it is the real part of a holomorphic function. I am unsure what a generalization of that property could be.

Proof of Proposition 3.10. (i) is trivial by definition of pluriharmonicity. (ii) follows instantly from Proposition 3.5. For (iii): if f is (anti)holomorphic, then $\mathrm{d}^c f = \mathrm{d}f \circ J_M = \pm J_N \circ \mathrm{d}f$. Since N is Kähler, $\nabla J_N = 0$ (one can show that $\nabla J = 0$ if and only if J is integrable and $\mathrm{d}\omega = 0$ in an almost Hermitian manifold). It easily follows that $\mathrm{d}_\nabla(J_N \circ \alpha) = J_N \circ \mathrm{d}_\nabla(\alpha)$ for any $\alpha \in \Omega^1(M, f^*\mathrm{T}N)$. In particular, $\mathrm{d}_\nabla \mathrm{d}^c f = \pm\, \mathrm{d}_\nabla(J_N \circ \mathrm{d}f) = \pm J_N \circ \mathrm{d}_\nabla \mathrm{d}f$. Recall that $\mathrm{d}_\nabla \mathrm{d}f = 0$ is always true (Proposition 1.10), therefore we find $\mathrm{d}_\nabla \mathrm{d}^c f = 0$, i.e. f is pluriharmonic. □

Remark 3.12. As Nicolas Tholozan pointed out to us, it is not hard to argue that the converse of Proposition 3.10 (ii) is also true locally: if f is harmonic with respect to any locally defined Kähler metric, then f is pluriharmonic.

Example 3.13. As a special case of Proposition 3.10 (iii), if N is Kähler manifold and M is a complex submanifold, then the embedding of M in N is minimal (i.e. it is a harmonic isometric immersion). For example, consider complex submanifolds $\mathbb{CP}^1 \to \mathbb{CP}^3$ induced by linear subspaces $\mathbb{C}^2 \to \mathbb{C}^4$. Such embeddings of $\mathbb{CP}^1 \approx S^2$ in $\mathbb{CP}^3 \approx S^4$ thus are minimal spheres.

For more properties and results on pluriharmonic maps, refer to [OU90].

4 The Siu–Sampson theorem

The main goal of this section is to explain the Siu–Sampson theorem below (Theorem 4.1), first proven by Sampson [Sam86][1] following the Bochner technique of Siu [Siu80].

Theorem 4.1 (Siu, Sampson)**.** *Let M be a compact Kähler manifold and let N be Riemannian manifold of nonpositive Hermitian sectional curvature. If $f\colon M \to N$ is harmonic, then f is pluriharmonic. Moreover:*

$$R^N\left(X, Y, \bar{X}, \bar{Y}\right) = 0 \tag{19}$$

for all $x \in M$ and for all $X, Y \in \mathrm{d}f(\mathrm{T}^{1,0}_x M)$.

In (19), R^N is the complexification of the Riemann curvature tensor of N to $\mathrm{T}N \otimes \mathbb{C}$. By definition, N has nonpositive Hermitian sectional curvature if $R^N(X, Y, \bar{X}, \bar{Y}) \leqslant 0$ for all $y \in N$ and $X, Y \in \mathrm{T}_y N$: more on this in § 4.1.

4.1 Strong curvature conditions

4.1.1 Hermitian sectional curvature and strongly nonpositive curvature

The Siu–Sampson Theorem 4.1 requires the notion of Hermitian sectional curvature. Contrary to what its name might suggest, Hermitian sectional curvature does not require a complex structure on the manifold, it is a purely Riemannian notion.

Let N be a Riemannian manifold. Please refer to § 2.3.1 for the definitions of the standard curvature tensors on N: the Riemann tensor R and its purely covariant version, the curvature operator Q, and sectional curvature K. We still denote R and Q the complex linear extensions of these operators to complexified vectors, e.g. $Q\colon \otimes^2(\mathrm{T}N \otimes \mathbb{C}) \times \otimes^2(\mathrm{T}N \otimes \mathbb{C}) \to \mathbb{C}$.

We say that *N has nonpositive Hermitian sectional curvature* if $R(X, Y, \bar{X}, \bar{Y}) \leqslant 0$ for any $X, Y \in \mathrm{T}N \otimes \mathbb{C}$. Following the terminology of Siu [Siu80], we alternatively say that *N has strongly nonpositive curvature*.

Clearly, N has strongly nonpositive curvature if and only if $Q(\sigma, \bar{\sigma}) \leqslant 0$ for any decomposable tensor $\sigma \in \otimes^2(\mathrm{T}N \otimes \mathbb{C})$. If moreover $Q(\sigma, \bar{\sigma}) \leqslant 0$ on all $\otimes^2(\mathrm{T}N \otimes \mathbb{C})$, we say that *$N$ has very strongly nonpositive curvature*. This is equivalent to $Q(\sigma, \sigma) \leqslant 0$ on $\otimes^2 \mathrm{T}N$, in other words Q is negative semidefinite on $\otimes^2 \mathrm{T}N$.

We have obvious analogous notions of (very) strongly nonnegative curvature.

Example 4.2. A manifold of strongly nonpositive curvature clearly has nonpositive sectional curvature. It is easy to see that the converse holds for a manifold of constant sectional curvature. More generally it holds for a manifold N of negative δ-pinched curvature with $\delta \geqslant 1/4$ by [YZ91]. This means that for any $x \in N$, the sectional curvature at x is bounded between $-c_x$ and $-\delta c_x$ for some constant $c_x > 0$ (δ is always assumed in $(0, 1]$). It is claimed e.g. in [ABC+96] or [Zhe00] that the converse is false in general, but we are not aware of a counter-example.

Example 4.3. By Jost-Yau [JY83], a product of hyperbolic spaces $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ has strongly nonpositive curvature. We believe that it actually has very strongly nonpositive curvature, in fact we do not know an example of manifold with strongly, but not very strongly, nonpositive curvature.

[1] See Theorem 1 p. 129: Although Sampson does not use the terminology *pluriharmonic*, the equations $y^j_{\alpha|\bar{\beta}} = 0$ mean that $\mathrm{d}^{0,1}_\nabla \mathrm{d}^{1,0} f = 0$, i.e. f is pluriharmonic by Proposition 3.8.

Example 4.4. Symmetric spaces of noncompact type have very strongly nonpositive curvature (see § 5). Similarly, symmetric spaces of compact type have very strongly nonnegative curvature.

Example 4.5. Y. Wu [Wu14] showed that the Teichmüller space of a closed surface of genus > 1 has very strongly nonpositive curvature.

4.1.2 Strongly negative curvature of Kähler manifolds

The natural attempt to define, for example, very strongly negative curvature by requiring that $Q(\sigma, \bar{\sigma}) < 0$ for all nonzero $\sigma \in \otimes^2(\mathrm{T}N \otimes \mathbb{C})$ turns out to not be reasonable. Indeed, assume that N is a Kähler manifold. It follows immediately from Proposition 3.4 that $Q(\sigma, \bar{\sigma}) = 0$ for any σ of type $(2, 0)$ or of type $(0, 2)$, e.g. $\sigma = X \otimes Y$ with $X, Y \in \mathrm{T}^{1,0}N$.

By definition, a Kähler manifold N has *strongly negative curvature* if $Q(\sigma, \bar{\sigma}) < 0$ for all nonzero $\sigma \in \mathrm{T}^{1,0}N \otimes \mathrm{T}^{0,1}N$ of length $\leqslant 2$, e.g. $\sigma = X \otimes \bar{Y} + Z \otimes \bar{W}$, and N has *very strongly negative curvature* if $Q(\sigma, \bar{\sigma}) < 0$ for all nonzero $\sigma \in \mathrm{T}^{1,0}N \otimes \mathrm{T}^{0,1}N$. (In other words, N has very strongly negative curvature if $Q^{1,1}$ is negative definite as a Hermitian-symmetric form on $\Lambda^{1,1}\,\mathrm{T}N$.)

Example 4.6. Siu [Siu80] proved that complex hyperbolic space $\mathbb{C}H^n$ has very strongly negative curvature. Complex projective space $\mathbb{C}P^n$ has very strongly positive curvature. Mostow-Siu [MS80] provided an example of compact Kähler manifold with very strongly negative curvature that is not a quotient of $\mathbb{C}H^n$.

Example 4.7. According to [Zhe00, Theorem 9.26], a Kähler manifold of complex dimension 2 has very strong negative (resp. nonpositive) curvature if and only if it has negative (resp. nonpositive) sectional curvature. We expect this to be false in higher dimensions, but we are not aware of a counter-example.

4.1.3 Non-example: the Penrose twistor fibration

Let us illustrate with a non-example that the curvature condition is important for the Siu–Sampson theorem Theorem 4.1. Consider the map (called *Penrose twistor fibration*, see e.g. [Bry85])

$$f \colon \mathbb{C}P^3 = \{\mathbb{C}\text{-lines in } \mathbb{C}^4\} \longrightarrow \mathbb{H}P^1 = \{\mathbb{H}\text{-lines in } \mathbb{H}^2\}$$

defined by $f(l) = L$, where L is the unique $\mathbb{H}$-line in $\mathbb{H}^2 \approx \mathbb{C}^4$ containing l. Here $\mathbb{H}$ denotes the $\mathbb{R}$-algebra of quaternions. Topologically, $\mathbb{H}P^1 \approx S^7/S^3 \approx S^4$, and $\mathbb{H}P^1$ has a natural Riemannian (in fact quaternion-Kähler) metric of constant curvature 1, making the Hopf fibration $S^3 \to S^7 \to S^4$ a Riemannian submersion. In particular $\mathbb{H}P^1$ does not have nonpositive Hermitian sectional curvature.

We claim that f is harmonic. Indeed, one can check that f is a Riemannian submersion, and it is known since [ES64] that a Riemannian submersion is harmonic if and only if its fibers are minimal submanifolds. Here the fibers of f are minimal spheres as we have seen in Example 3.13.

However f is not pluriharmonic. Indeed, consider a fiber C of f and a transverse $\mathbb{C}$-plane $C' \subseteq \mathbb{C}^4$. For any $\varepsilon > 0$, it is possible to choose C' contained in the ε-neighborhood of C in $\mathbb{C}P^3$ (just tilt C slightly) and such that C' is not a fiber of f. Then the restrition of f to C' is nonconstant and maps C' into the ε-neighborhood a point (namely $f(C)$) in $\mathbb{H}P^1 \approx S^4$. Such a map cannot be harmonic byvthe maximum principle. Since C' is a complex curve in $\mathbb{C}\mathrm{P}^3$, f is not pluriharmonic.

For the non-experts, we explain why any harmonic map from a closed manifold M to a Riemannian manifold N whose image is contained in a small enough ball $B(p, \varepsilon)$ must be constant.

For ε small enough, the distance squared function $\varphi = d(p, \cdot)^2$ is convex on $B(p, \varepsilon)$, i.e. it has positive semidefinite Hessian. For example, on a round sphere S^n, one can take an open hemisphere for $B(p, \varepsilon)$. However, harmonic functions pull back convex functions to subharmonic functions. Indeed, one has the composition formula

$$\Delta(\varphi \circ f) = \mathrm{d}\varphi \circ \Delta(f) + \operatorname{tr}\left((\nabla^2 \varphi)(\mathrm{d}f, \mathrm{d}f)\right)$$

so if $\Delta f = 0$, $\Delta(\varphi \circ f) \geqslant 0$ when φ is convex. (The converse is true: f is harmonic if it pulls back any locally defined convex function to a subharmonic function [Ish79, Thm 3.4].) This implies that any local maximum of $\varphi \circ f$ cannot be attained at an interior point of M unless f is constant by the maximum principle for subharmonic functions. By compactness of M, $\varphi \circ f$ does attain a global maximum, however M only has interior points (no boundary), so f must be constant.

4.2 Sampson's Bochner formula

Siu [Siu80] was the first to find a Bochner formula (also known as *Siu's* $\partial\bar{\partial}$*-trick*) for maps between Kähler manifolds and obtain rigidity results. Sampson [Sam86] gave an improvement of Siu's Bochner formula that implies Theorem 4.1.

Let us explain Sampson's formula. (The reader may also refer to [Xin96, Eq. (4.35)] for a similar exposition.) First recall that the *divergence* of a tensor field $T \in \Gamma(\mathrm{T}^* M \otimes \dots)$ is given by

$$\begin{aligned} \operatorname{div} T &= \operatorname{tr} \nabla T \\ &= \sum_{j=1}^{m} (\nabla_{e_j} T)(e_j, \dots) \end{aligned} \tag{20}$$

where $(e_j)_{1 \leqslant j \leqslant m}$ is any local orthonormal frame field. This definition makes sense for $T \in \Gamma(\mathrm{T}^* M \otimes E)$ where E is any vector bundle with a connection: take the tensor product connection in (20). Note that, seeing T as a 1-form with values in E, (10) shows that $\mathrm{d}_\nabla^* T = -\operatorname{div}(T)$.

Let $f \colon (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds. Define the first fundamental form $\theta := f^* g_N \in \Gamma(\mathrm{T}^* M \otimes \mathrm{T}^* M)$. Note that by definition, $\|\mathrm{d}f\|^2 = \operatorname{tr}\theta$. Taking the divergence of θ, one finds

$$\operatorname{div}(\theta) = \langle \Delta f, \mathrm{d}f(\cdot)\rangle + \sum_{j=1}^{m} \langle \mathrm{d}f(e_j), (\nabla_{e_j} \mathrm{d}f)(\cdot)\rangle \,.$$

This is an element of $\Gamma(\mathrm{T}^* M)$. One can take the divergence again and using the Weitzenböck formula (15) one readily finds:

$$\begin{aligned} \operatorname{div}^2(\theta) = {}& 2\langle \nabla \Delta f, \nabla f\rangle + \|\Delta f\|^2 + \|\nabla^2 f\|^2 \\ &+ \operatorname{Ric}^M(\langle \nabla f, \nabla f\rangle_N) - R^N(\langle \nabla f \wedge \nabla f\rangle_M, \langle \nabla f \wedge \nabla f\rangle_M)\,. \end{aligned} \tag{21}$$

It is easy to see that (21) is merely a rewriting of the Bochner formula (17) by first realizing that $\operatorname{div}\theta - \frac{1}{2}\mathrm{d}\|\mathrm{d}f\|^2 = \langle \mathrm{d}f, \nabla \Delta f\rangle$.

Sampson's computation is based on the observation that when (M, J) is a complex manifold, given a bilinear form $B \in \Gamma(\mathrm{T}^* M \otimes \mathrm{T}^* M \otimes \dots)$, as an alternative to the trace $\operatorname{tr} B$, one can define

$$\operatorname{tr}^c(B) = \sum_{i=1}^{M} B(\bar{\varepsilon}_j, \varepsilon_j)$$

where $\varepsilon_j = {e_j}^{1,0}$. It is immediate to check that $\operatorname{tr}^c B = \frac{1}{4}\left[\operatorname{tr} B + i \operatorname{tr}\left(B(J\cdot,\cdot)^{1,1}\right)\right]$ and that if B is symmetric then $\operatorname{tr}^c(B) = \frac{1}{4}\operatorname{tr} B$. In particular one can define $\operatorname{div}^c(T) = \operatorname{tr}^c(\nabla T)$.

Reproducing the calculations leading to (21) by computing $(\operatorname{div}^c)^2(\theta)$ instead of $\operatorname{div}^2(\theta)$, one first finds:

$$\operatorname{div}^c \theta = \langle \operatorname{tr}^c(\nabla^2 f), \mathrm{d}f(\cdot)\rangle + \langle \mathrm{d}f(\varepsilon_l), (\nabla_{\bar{\varepsilon}_l} \mathrm{d}f)(\cdot)\rangle \ . \tag{22}$$

Be advised that in (22) we omit the summation symbol over the repeated index j; we maintain this convention in the remainder of this section. Since $\nabla \mathrm{d}f$ is symmetric (Proposition 1.2), $\operatorname{tr}^c(\nabla^2 f) = \frac{1}{4}\operatorname{tr}(\nabla^2 f) = \frac{1}{4}\Delta f$. In particular, if f is harmonic, (22) reduces to

$$\operatorname{div}^c \theta = \langle \mathrm{d}f(\varepsilon_l), (\nabla_{\bar{\varepsilon}_l} \mathrm{d}f)(\cdot)\rangle \ .$$

Taking div^c again and using the Weitzenböck formula (15), one finds:

$$\begin{aligned}(\operatorname{div}^c)^2\,\theta = {}& \langle \nabla_{\bar{\varepsilon}_j} \mathrm{d}f(\varepsilon_l), \nabla_{\bar{\varepsilon}_l} \mathrm{d}f(\varepsilon_j)\rangle + \langle \mathrm{d}f(R^M(\bar{\varepsilon}_j, \bar{\varepsilon}_l)\varepsilon_j), \mathrm{d}f(\varepsilon_l)\rangle \\ & - R^N(\mathrm{d}f(\bar{\varepsilon}_j), \mathrm{d}f(\bar{\varepsilon}_l), \mathrm{d}f(\varepsilon_j), \mathrm{d}f(\varepsilon_l)) \ .\end{aligned}$$

If M is Kähler, then the curvature term $\langle \mathrm{d}f(R^M(\bar{\varepsilon}_j, \bar{\varepsilon}_l)\varepsilon_j), \mathrm{d}f(\varepsilon_l)\rangle$ vanishes. This is straightforward to show noticing that this term is real (unchanged by complex conjugation by symmetry of the Riemann curvature tensor) and recalling that the Ricci curvature tensor of M is of type $(1,1)$ (Proposition 3.4). Sampson's Bochner formula follows:

Proposition 4.8 (Sampson's Bochner formula for harmonic maps)**.** *Let $f\colon M \to N$ be a smooth harmonic map where M is a Kähler manifold and N is a Riemannian manifold. Then*

$$(\operatorname{div}^c)^2\,\theta = \langle \nabla_{\bar{\varepsilon}_j} \mathrm{d}f(\varepsilon_l), \nabla_{\bar{\varepsilon}_l} \mathrm{d}f(\varepsilon_j)\rangle - R^N(\mathrm{d}f(\bar{\varepsilon}_j), \mathrm{d}f(\bar{\varepsilon}_l), \mathrm{d}f(\varepsilon_j), \mathrm{d}f(\varepsilon_l)) \ . \tag{23}$$

It is remarkable that this Bochner formula does not involve the curvature of M.

Remark 4.9. It is easy to generalize Sampson's Bochner formula to any smooth map without the harmonicity condition by keeping track of the extra terms:

$$\begin{aligned}(\operatorname{div}^c)^2\,\theta = {}& 2\langle \nabla_{\bar{\varepsilon}_j}(\nabla_{\bar{\varepsilon}_l} \mathrm{d}f(\varepsilon_l)), \mathrm{d}f(\varepsilon_j)\rangle + \langle \nabla_{\bar{\varepsilon}_l} \mathrm{d}f(\varepsilon_l), \nabla_{\bar{\varepsilon}_j} \mathrm{d}f(\varepsilon_j)\rangle \\ & + \langle \nabla_{\bar{\varepsilon}_j} \mathrm{d}f(\varepsilon_l), \nabla_{\bar{\varepsilon}_l} \mathrm{d}f(\varepsilon_j)\rangle - R^N(\mathrm{d}f(\bar{\varepsilon}_j), \mathrm{d}f(\bar{\varepsilon}_l), \mathrm{d}f(\varepsilon_j), \mathrm{d}f(\varepsilon_l)) \ .\end{aligned} \tag{24}$$

In [OV90], Ohnita-Valli give a neat variant of Sampson's Bochner formula (24), although they do not give any details for the proof. First define the *energy form* of $f\colon M \to N$ by $\varepsilon(f) \coloneqq (f^* g_N(J\cdot,\cdot))^{(1,1)}$. This is a finer version of the energy density of f: the two are related by $e(f)\operatorname{vol}_M = \varepsilon(f) \wedge \frac{\omega^{n-1}}{(n-1)!}$ where $n = \dim_{\mathbb{C}} M$.

Proposition 4.10 ([OV90, Eq. (1.3)])**.** *Let $f\colon M \to N$ be a smooth map where M is a Kähler manifold and N is a Riemannian manifold. Then*

$$\begin{aligned}i\partial\bar{\partial}\varepsilon(f) \wedge \frac{\omega^{n-2}}{(n-2)!} = {}& \Big[\|\nabla^{0,1}\mathrm{d}^{1,0} f\|^2 - \|\operatorname{tr}(\nabla^{0,1}\mathrm{d}^{1,0} f)\|^2 \\ & - R^N(\mathrm{d}f(\varepsilon_j), \mathrm{d}f(\varepsilon_l), \mathrm{d}f(\bar{\varepsilon}_j), \mathrm{d}f(\bar{\varepsilon}_l))\Big] \operatorname{vol}_M \ .\end{aligned} \tag{25}$$

4.3 Proof of the Siu–Sampson theorem

4.3.1 Using Sampson's Bochner formula

Just like Eells-Sampson's rigidity Theorem 2.2, the Siu–Sampson Theorem 4.1 is simply obtained by integrating the Bochner formula over M, arguing that the left-hand side has zero integral, and that all terms on the right-hand side are pointwise $\geqslant 0$ and hence must vanish everywhere.

However Sampson [OV90] (also [CT89] and [Xin96, Eq. (4.35)]) concludes perhaps too quickly from the Bochner formula (23): it is not clear (to us at least!) that $(\operatorname{div}^c)^2\,\theta$ (or either side of the equality) is co-closed, or even real.

That being said, Theorem 2.2 does follow immediately from the Ohnita-Valli version of the Bochner formula (25). Indeed, observe that:

- It is always the case that $i\partial\bar{\partial} = \frac{1}{2}\,\mathrm{d}\,\mathrm{d}^{\mathrm{c}}$ on $\mathbb{C}$-valued differential forms. In particular, $i\partial\bar{\partial}\varepsilon(f)$ is closed, and so is $i\partial\bar{\partial}\varepsilon(f) \wedge \frac{\omega^{n-2}}{(n-2)!}$ since $\mathrm{d}\omega = 0$ (i.e. M is Kähler).
- Recall that $\mathrm{d}_\nabla \mathrm{d}^{\mathrm{c}} f = 2i\, \mathrm{d}_\nabla^{0,1}\, \mathrm{d}^{1,0} f$, in particular $\|\nabla^{0,1}\,\mathrm{d}^{1,0} f\|^2 = 0$ if and only if f is pluriharmonic.
- Since $\nabla^2 f$ is symmetric, $\operatorname{tr}(\nabla^{0,1}\,\mathrm{d}^{1,0} f) = \operatorname{tr}^c(\nabla^2 f) = \frac{1}{4}\operatorname{tr}(\nabla^2 f) = \frac{1}{4}\Delta f$. In particular $\|\operatorname{tr}(\nabla^{0,1}\,\mathrm{d}^{1,0} f)\|^2 = 0$ if and only if f is harmonic.

4.3.2 Toledo's proof

As an alternative to Sampson's perhaps slightly dubitable argument, an elegant proof was written by D. Toledo in Chapter 6 of the excellent book [ABC+96]. Let us briefly explain this argument which does not require a Bochner formula beforehand.

Proof of Theorem 2.2. First assume for simplicity that $N = \mathbb{R}$. Let M be a Kähler manifold and let $f\colon M \to \mathbb{R}$ be harmonic, i.e. $\mathrm{d}\,\mathrm{d}^{\mathrm{c}} f \wedge \omega^{n-1} = 0$ by Proposition 3.5. We want to show that f is pluriharmonic i.e. $\mathrm{d}\,\mathrm{d}^{\mathrm{c}} f = 0$ (Proposition 3.8). Consider the form $\eta \in \Omega^{2n}(M,\mathbb{R})$ where $n = \dim_{\mathbb{C}} M$ defined by

$$\eta = \mathrm{d}\,\mathrm{d}^{\mathrm{c}} f \wedge \mathrm{d}\,\mathrm{d}^{\mathrm{c}} f \wedge \omega^{n-2}\,.$$

By the linear algebra Lemma 4.12 whose proof we postpone, η is pointwise $\leqslant 0$ with equality if and only if $\mathrm{d}\,\mathrm{d}^{\mathrm{c}} f = 0$. On the other hand η is closed since $\eta = \mathrm{d}\left(\mathrm{d}^{\mathrm{c}} f \wedge \mathrm{d}\,\mathrm{d}^{\mathrm{c}} f \wedge \omega^{n-2}\right)$ (recall that $\mathrm{d}\omega = 0$ since M is assumed Kähler), so $\int_M \eta = 0$ by Stokes' theorem. We conclude that η must vanish everywhere, as must $\mathrm{d}\,\mathrm{d}^{\mathrm{c}} f$.

For the general case where N is any Riemannian manifold with nonpositive Hermitian sectional curvature, define $\eta \in \Omega^{2n}(M,\mathbb{R})$ by

$$\begin{aligned}\eta &= \mathrm{d}\left(\langle \mathrm{d}_\nabla \mathrm{d}^{\mathrm{c}} f \wedge \mathrm{d}^{\mathrm{c}} f\rangle \wedge \omega^{n-2}\right)\\ &= \langle \mathrm{d}_\nabla^2\, \mathrm{d}^{\mathrm{c}} f \wedge \mathrm{d}^{\mathrm{c}} f\rangle \wedge \omega^{n-2} + \langle \mathrm{d}_\nabla \mathrm{d}^{\mathrm{c}} f \wedge \mathrm{d}_\nabla \mathrm{d}^{\mathrm{c}} f\rangle \wedge \omega^{n-2}\,.\end{aligned} \tag{26}$$

By definition η is closed so we have again $\int_M \eta = 0$. A straightforward extension of the linear algebra Lemma 4.12 shows that just as in the $N = \mathbb{R}$ case, the term $\langle \mathrm{d}_\nabla \mathrm{d}^{\mathrm{c}} f \wedge \mathrm{d}_\nabla \mathrm{d}^{\mathrm{c}} f\rangle \wedge \omega^{n-2}$ is pointwise $\leqslant 0$, with equality if and only if $\mathrm{d}_\nabla \mathrm{d}^c f = 0$.

It remains to discuss the curvature term $\langle \mathrm{d}_\nabla^2\, \mathrm{d}^{\mathrm{c}} f \wedge \mathrm{d}^{\mathrm{c}} f\rangle \wedge \omega^{n-2}$. Let us skip some details and instead cite Toledo [Tol99]: *When rewritten using the definition of curvature* $\mathrm{d}_\nabla^2 = -R$*,* *this term turns out to be the average value of $R^N(\mathrm{d}f(X), \mathrm{d}f(Y), \mathrm{d}f(\bar{X}), \mathrm{d}f(\bar{Y})$ over all unit length decomposable vectors $X \wedge Y \in \otimes^2 \mathrm{T}^{1,0}M$. This computation can be found in [ABC+96] or in equivalent forms in [Siu80, Sam86].* The conclusion immediately follows from this claim. □

Remark 4.11. For the record, we find that even in the detailed computation of [ABC+96], some important arguments are overlooked: when Toledo writes $\mathrm{d}^2_\nabla = -R$, the curvature tensor R should not just be the Riemann curvature tensor of N, but also involve the curvature tensor of M (see § 2.3.2). As we explained in § 4.1, the fact that R^M ends up not playing a role is a little miracle due to the fact that the Ricci tensor of a Kähler manifold is of type $(1, 1)$.

To complete the proof we need the following linear algebra lemma, which can be seen as a special case of the *Hodge-Riemann bilinear relations*, a well-known result in Kähler geometry that leads to the Hodge index theorem (see e.g. [Voi08]).

Lemma 4.12. *Let (V, g, J, ω) be a Hermitian vector space of complex dimension n. If $\alpha \in \Lambda^{1,1}V^*$ is a $(1, 1)$-form such that $\alpha \wedge \omega^{n-1} = 0$, then $\alpha \wedge \alpha \wedge \omega^{n-2} \leqslant 0$, with equality if and only if $\alpha = 0$.*

For the reader's (but mostly the author's) entertainment, we give a neat proof (mostly taken from [ABC+96]) of this not so simple lemma.

Proof. Recall that the Hermitian metric in V is $h := g - i\omega$, the space of complex-linear endomorphisms is is $\mathrm{End}_{\mathbb{C}}(V) := \{f \in \mathrm{End}_{\mathbb{R}}(V) \mid fJ = Jf\}$, the space of Hermitian self-adjoint endomorphisms is $\mathcal{H}(V) := \{f \in \mathrm{End}_{\mathbb{C}}(V) \mid h(f(x), y) = h(x, f(y))\}$ and the unitary group is $\mathrm{U}(V) := \{u \in \mathrm{End}_{\mathbb{C}}(V) \mid h(u(x), u(y)) = h(x, y)\}$.

An element $\alpha \in \Lambda^{1,1}V^*$ is a skew-symmetric bilinear form on V such that $\alpha(Jx, Jy) = \alpha(x, y)$. We define an isomorphism

$$\begin{aligned} \mathcal{H}(V) &\to \Lambda^{1,1}V^* \\ f &\mapsto \omega_f \end{aligned} \tag{27}$$

where $\omega_f(x, y) := \omega(fx, y)$. This isomorphism is moreover $\mathrm{U}(V)$-equivariant for the natural action of $\mathrm{U}(V)$ on $\mathcal{H}(V)$ by conjugation and on $\Lambda^{1,1}V^*$ by pullback: $(u \cdot \alpha)(x, y) = \alpha(ux, uy)$.

We claim that the decomposition of $\mathcal{H}(V)$ as a sum of irreducible $\mathrm{U}(V)$-modules is $\mathcal{H} = \mathbb{R}id + \mathcal{H}^0$, where $\mathcal{H}^0$ is the space of traceless self-adjoint endomorphisms. Indeed, consider the Cartan decomposition $\mathfrak{sl}(V) = \mathfrak{su}(h) \oplus \mathfrak{p}$, where $\mathfrak{su}(V)$ is the Lie algebra of $\mathrm{SU}(V)$ and $\mathfrak{p} = \mathcal{H}^0$. If there existed a nonzero proper $\mathrm{U}(h)$-invariant subspace $\mathfrak{m} \subseteq \mathfrak{p}$, then $\mathfrak{u} + \mathfrak{m}$ would be a nonzero proper ideal of the Lie algebra $\mathfrak{su}(V)$, contradicting its well-known simplicity.

Via the $\mathrm{U}(h)$-equivariant isomorphism (27), we get a decomposition of $\Lambda^{1,1}V^*$ into irreducible $\mathrm{U}(h)$-modules as

$$\Lambda^{1,1}V^* = \mathbb{R}\omega \oplus \Lambda^{1,1}_0 V^*$$

where $\alpha \in \Lambda^{1,1}_0 V^*$ if and only if $\alpha \wedge \omega^{n-1} = 0$.

To conclude, consider the inner product

$$\begin{aligned} \Lambda^{1,1}V^* \times \Lambda^{1,1}V^* &\to \Lambda^{2n}V^* \approx \mathbb{R} \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta \wedge \omega^{n-2}\ . \end{aligned} \tag{28}$$

This inner product is $\mathrm{U}(V)$-invariant, therefore in restriction to the irreducible subspace $\Lambda^{1,1}_0 V^*$ it must be either positive definite or negative definite or identically zero. Compute any example to conclude that it is negative definite. □

Remark 4.13. Toledo's proof is very closely related to the Ohnita-Valli version of Sampson's Bochner formula (25). Indeed, it is easy to check that $\varepsilon(f) = -\frac{1}{2}\langle \mathrm{d}f \wedge \mathrm{d}^c f\rangle$ and derive that

$$\eta = -4i\partial\bar{\partial}\varepsilon(f) \wedge \omega^{n-2}\,.$$

One can therefore equate Toledo's identity (26) with the Ohnita-Valli formula (25) as long as one can equate the curvature terms, which is done in [ABC+96], and upgrade Lemma 4.12: the inner product (28) is actually given explicitly by

$$\alpha \wedge \alpha \wedge \frac{\omega^{n-2}}{(n-2)!} = (\operatorname{tr}\alpha)^2 - \|\alpha\|^2\,.$$

We do not know of an elegant proof of this identity, but it can be shown by brute force, taking an orthonormal basis $(e_1, Je_1, \dots)$ of V, etc.

4.4 First applications to rigidity

This first straightforward consequence of Theorem 4.1 is due to Sampson [Sam86]:

Theorem 4.14. *Let $f\colon M \to N$ where M is compact Kähler and N has constant negative sectional curvature. If f is harmonic, then the rank of f is everywhere $\leqslant 2$.*

Proof. If N has constant sectional curvature k, then $R^N(X,Y,\bar{X},\bar{Y}) = k\|X \wedge Y\|^2$ for any $X,Y \in \mathrm{T}N \otimes \mathbb{C}$. By Theorem 4.1, if f is harmonic then $\mathrm{d}f(X^{1,0})$ and $\mathrm{d}f(Y^{1,0})$ must be collinear for any $X,Y \in \mathrm{T}M$. This implies that $\mathrm{rk}(f) \leqslant 2$. ◻

Next, the strong rigidity theorem of Siu for strongly negatively curved Kähler manifolds:

Theorem 4.15. *Let M and N be compact Kähler manifolds, with N strongly negatively curved. Any harmonic map $M \to N$ is holomorphic or antiholomorphic unless it has rank $\leqslant 2$ everywhere.*

Corollary 4.16. *Let N be a compact Kähler manifolds of complex dimension at least 2 with strongly negative curvature. Then any compact Kähler manifold homotopy equivalent to N must be either biholomorphic or bi-antiholomorphic to N.*

Corollary 4.16 says that N is *strongly rigid* as a Kähler manifold. This echoes the celebrated rigidity theorem of Mostow, more about this in § 5.

Proof of Theorem 4.15 and Corollary 4.16. If f is neither holomorphic nor antiholomorphic at x, then there exist $X_1, X_2 \in \mathrm{T}^{1,0}_x M$ such that $\mathrm{d}f(X_1)$ is not of type $(1,0)$ and $\mathrm{d}f(X_2)$ is not of type $(0,1)$. One can then find $X_0 \in \mathrm{T}^{1,0}_x M$ such that $\mathrm{d}f(X_0)$ is of mixed type (either $X_0 = X_1 + X_2$ or $X_0 = X_2$ works). Given any $Y \in \mathrm{T}^{1,0}_x M$, consider $\sigma := \mathrm{d}f(X_0)^{1,0} \otimes \overline{\mathrm{d}f(Y)^{0,1}} - \mathrm{d}f(Y)^{1,0} \otimes \mathrm{d}f(X_0)^{0,1}$. If f is harmonic, then by Theorem 4.1 $R(\mathrm{d}f(X_0), \mathrm{d}f(Y), \overline{\mathrm{d}f(X_0)}, \overline{\mathrm{d}f(Y)}) = Q(\sigma, \bar{\sigma}) = 0$. By strongly negative curvature of N, it follows $\sigma = 0$. This implies that $\mathrm{d}f(Y)$ and $\mathrm{d}f(X_0)$ are collinear over $\mathbb{C}$, and since this is true for all Y, the rank of f at x is $\leqslant 2$. If f is not holomorphic (resp. antiholomorphic) on M, the set U (resp. V) where $\mathrm{d}f(\cdot)^{0,1} \neq 0$ (resp. $\mathrm{d}f(\cdot)^{1,0} \neq 0$) is open dense. Thus f must have rank $\leqslant 2$ on the open dense set $U \cap V$, and hence everywhere.

For Corollary 4.16, let $f\colon M \to N$ be a homotopy equivalence. By the Eells-Sampson Theorem 2.3 we may assume f is harmonic. Since M and N are closed, they must have same dimension and f has degree 1. In particular f is surjective and has full rank $n \leqslant 4$ somewhere. By Theorem 4.15, f is holomorphic or antiholomorphic. If f is not injective, then it must send a subvariety $Z \subseteq M$ of positive dimension to a point. This is not possible because the fundamental class of Z is a nontrivial cohomology class of M (the integral of a power of the Kähler form on Z is the volume of Z, hence positive) and f induces an isomorphism on cohomology. □

For further rigidity-type results revolving around harmonic maps and complex manifolds we refer to [EL95a] and [OU90].

5 Applications to symmetric spaces

5.1 Symmetric spaces of noncompact type

Let $X = G/K$ be a symmetric space of noncompact type: G is a semisimple Lie group without compact factors and K is a maximal compact subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively, and denote B the Killing form on $\mathfrak{g}$. Recall that B is an $\operatorname{Ad} G$-invariant symmetric bilinear form on $\mathfrak{g}$, and it is nondegenerate because $\mathfrak{g}$ is semisimple.

We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{p}$ is the B-orthogonal of $\mathfrak{k}$. This decomposition satisfies $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Moreover B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Thus B induces an inner product on $\mathrm{T}_{Ke} X \approx \mathfrak{p}$, which can be uniquely extended as a G-invariant Riemannian metric on X.

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