Bi-Lagrangian structures and Teichmüller theory

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Abstract

This paper has two purposes: the first is to study several structures on manifolds in the general setting of real and complex differential geometry; the second is to apply this study to Teichmüller theory. We primarily focus on bi-Lagrangian structures, which are the data of a symplectic structure and a pair of transverse Lagrangian foliations, and are equivalent to para-Kähler structures. First we carefully study real and complex bi-Lagrangian structures and discuss other closely related structures and their interrelationships. Next we prove the existence of a canonical complex bi-Lagrangian structure in the complexification of any real-analytic Kähler manifold and showcase its properties. We later use this bi-Lagrangian structure to construct a natural almost hyper-Hermitian structure. We then specialize our study to moduli spaces of geometric structures on closed surfaces, which tend to have a rich symplectic structure. We show that some of the recognized geometric features of these moduli spaces are formal consequences of the general theory, while revealing other new geometric features. We also gain clarity on several well-known results of Teichmüller theory by deriving them from pure differential geometric machinery.

Key words and phrases: Bi-Lagrangian · para-Kähler · complex geometry · symplectic geometry of moduli spaces · Teichmüller theory · quasi-Fuchsian · hyper-Kähler

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Contents

Introduction

Let M be a smooth manifold. A symplectic structure on M is a closed nondegenerate 2-form ω. A Lagrangian foliation of (M, ω) is a foliation of M by Lagrangian submanifolds, *i.e.* submanifolds that are maximally isotropic for ω. Lagrangian foliations play a major role in geometric quantization where they are called real polarizations (see e.g. [BW97]). It was first proved by Weinstein [Wei71] that the leaves of a Lagrangian foliation admit a natural affine structure, which is given by a flat torsion-free affine connection called the Bott connection. This is only a *partial* connection on M in the sense that $\nabla_X Y$ is only defined for vector fields X and Y that are tangent to the foliation. This partial connection can always be extended to a "full" connection, but in general there is no canonical way to choose such an extension. However if (M, ω) is given *two* transverse Lagrangian foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, then there exists a unique symplectic connection ∇ on M which extends the Bott connection of each foliation (this was first proved by Heß [Heß80, Heß81], who studied symplectic connections in (bi-)polarized symplectic manifolds as a tool for geometric quantization). In this scenario $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is called a bi-Lagrangian structure on M and ∇ is called the bi-Lagrangian connection.

Bi-Lagrangian structures can alternatively be described as para-Kähler structures via a different approach that may seem more appealing to complex geometers, though one is quick to realize that the two points of view are equivalent. First note that on a smooth manifold M, the data of an ordered pair of distributions $L_1, L_2 \subset \mathrm{T}M$ of the same dimension such that $\mathrm{T}M = L_1 \oplus L_2$ is equivalent to an *almost para-complex* structure F, *i.e.* a traceless endomorphism of $\mathrm{T}M$ such that $F^2 = \mathbf{1}$ (indeed, such an endomorphism is determined by its ± 1-eigenspaces L_1 and L_2, which meet the requirements). Similarly to the case of an almost complex structure (recall that this is an endomorphism I of $\mathrm{T}M$ such that $I^2 = -\mathbf{1}$), there is a notion of integrability for an almost para-complex structure in terms of the vanishing of its Nijenhuis tensor, which amounts to the involutivity of both distributions L_1 and L_2 (*i.e.* stability under the Lie bracket). By the Frobenius theorem, this is equivalent to L_1 and L_2 being the tangent bundles to two transverse foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. In summary, a para-complex structure F is equivalent to an ordered pair of equidimensional transverse foliations $(\mathcal{F}_1, \mathcal{F}_2)$. In the presence of a symplectic structure ω, the condition that $\mathcal{F}_1$ and $\mathcal{F}_2$ are both Lagrangian is equivalent to the tensor field $g(u, v) := \omega(Fu, v)$ being symmetric, in which case it is a pseudo-Riemannian metric of signature (n, n), also called a neutral metric. The result of this discussion is that a bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is equivalent to a *para-Kähler structure* (g, F, ω), which is the para-complex analog of a Kähler structure (g, I, ω).

Thus, bi-Lagrangian structures lie at the intersection of symplectic, para-complex, and neutral pseudo-Riemannian geometry. They also relate to affine differential geometry via the Bott connections previously mentioned. Para-complex geometry is perhaps the least heavily investigated of these fields, although it has long been on mathematicians' radars ([Rs48], [Lib52a]). It shares many similarities with complex geometry, with the algebra of para-complex numbers playing the role of complex numbers (the imaginary unit squares to 1 instead of -1), with some important differences nonetheless. Interested readers may refer to [CFG96] and [Cor10, Chapter 15] for expositions on para-complex geometry.

In this paper we are interested in the natural analog of bi-Lagrangian structures in the complex (holomorphic) setting, which is essentially absent from the existing literature. Hence, we review complex symplectic structures and complex Lagrangian foliations, introduce complex Bott connections and complex bi-Lagrangian connections, etc., and study their basic properties. It is noteworthy that the algebra of bicomplex numbers naturally appears when one introduces para-complex structures in the complex setting. This is the algebra over the real numbers generated by three "units" i, j, f, such that $i^2 = j^2 = -f^2 = -1$ and $ij = ji = f$. The fact that any two of the three units are sufficient to generate this algebra enables several equivalent characterizations of complex bi-Lagrangian structures, encapsulated by a *bicomplex Kähler* structure.

In section 3 we focus on the complexification of a real-analytic Kähler manifold. We show that there is a canonical complex bi-Lagrangian structure there and that it has a nice description (Theorem 3.8). Let us recall that any real-analytic manifold N admits a complexification, *i.e.* an embedding in a complex manifold M as a maximally totally real submanifold. The germ of such a complexification is unique in the sense that any two complexifications agree in some neighborhood of N. When N is a complex manifold, there is a canonical model for its complexification which is $N^c = N \times \overline{N}$, where N is diagonally embedded. In particular, there is a canonical pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transverse ("vertical" and "horizontal") foliations in the complexification of a complex manifold, which is another way of saying that it has a natural bicomplex structure (cf. Theorem 3.12). When N is Kähler, the symplectic structure of N (locally) extends as a complex symplectic structure on N^c which refines $(\mathcal{F}_1, \mathcal{F}_2)$ into a complex bi-Lagrangian structure. This bi-Lagrangian structure can be identified to the bicomplex Kähler structure complexifying the Kähler structure of N (Theorem 3.13). After giving a quick application (Corollary 3.9 and Corollary 3.11), we conclude this section by explicitly working out the example of $\mathbb{C}P^1$ with the Fubini-Study metric (we describe in particular the affine structure in the canonical foliations of $\mathbb{C}P^1 \times \overline{\mathbb{C}P^1}$). Figure 1 below attempts to summarize in a diagram some of the interrelationships between all the structures discussed so far.

Delaying for the moment sections 4 and 5 on Teichmüller theory, in the final section (section 6) we push further the study of the complexification of a real-analytic Kähler manifold. We construct a natural almost hyper-Hermitian structure, relying on the complex bi-Lagrangian structure studied in section 3. Hyper-Hermitian structures are the quaternionic analog of Hermitian structures, and they are called hyper-Kähler when integrable. The Calabi metric [Cal79] in $\mathrm{T}^*\mathbb{C}P^n$ (previously discovered by Eguchi-Hanson [EH79] for $n = 1$) was the first nontrivial example of a hyper-Kähler structure on a noncompact manifold. Around the year 2000, Feix [Fei99] and Kaledin [VK99] independently proved that, more generally, there is a canonical hyper-Kähler structure in the cotangent bundle of any real-analytic Kähler manifold, although it is typically only defined in a neighborhood of the zero section. This hyper-Kähler structure can be naturally transported from the cotangent bundle to the complexification, although the resulting hyper-Kähler structure has never been properly characterized (cf. Question 6.8). We were initially hoping to recover the Feix-Kaledin hyper-Kähler structure in a more tangible way with our construction, but soon realized that our almost hyper-Hermitian structure is different as it is typically not integrable, even though it satisfies every other sensible requirement that one can ask of a "hyper-Kähler extension" of the Kähler structure (Theorem 6.10). It is nevertheless an interesting alternative; we show for instance that it is part of a richer *biquaternionic structure*, inducing in particular an almost para-quaternionic and para-hyper-Hermitian structure (Theorem 6.12 and Corollary 6.13). We conclude by working out the example of $\mathbb{C}P^1$ in detail.

Let us now turn to the applications to Teichmüller theory, which were the initial motivation of our work. Here we quote the first paragraph of the Foreword written by Papadopoulos–which we recommend reading *in extenso*–in the Handbook of Teichmüller theory [Pap07] :

> In a broad sense, Teichmüller theory is the study of moduli spaces for geometric structures on surfaces. This subject makes important connections between several areas in mathematics, including low-dimensional topology, hyperbolic geometry, dynamical systems theory, differential geometry, algebraic topology, representations of discrete groups in Lie groups, symplectic geometry, topological quantum field theory, string theory, and there are others.

Alongside differential and symplectic geometry, complex geometry and (pseudo-)Riemannian geometry can surely be appended to Papadopoulos' list; these features of Teichmüller theory are the focus of the present paper.

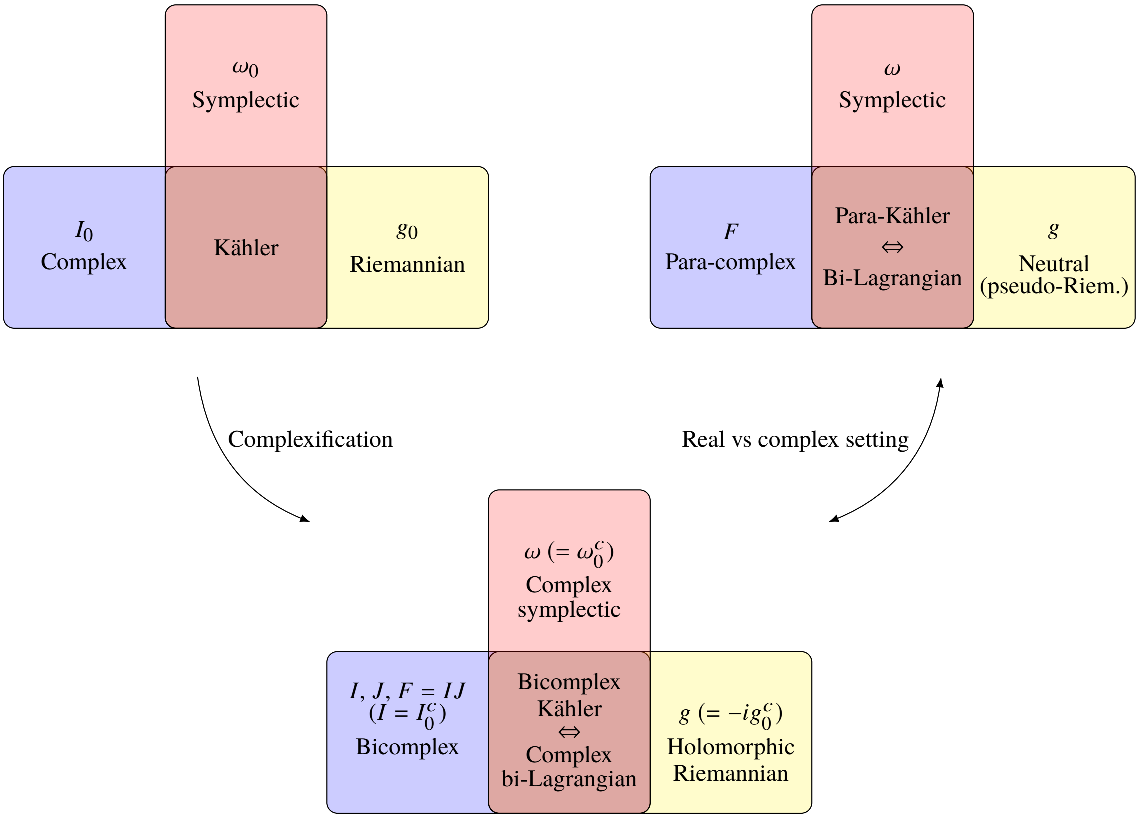

Figure 1: Partial overview of interrelationships between complex, para-complex, symplectic, and (pseudo-)Riemannian metric structures on real and complex manifolds.

Let S be a closed oriented surface of negative Euler characteristic. The Teichmüller space of S is the space of isotopy classes of complex structures on S. Teichmüller space and its complex-analytic structure were originally constructed and studied by Oswald Teichmüller in the 1930s. The naturality of its complex structure was fortified by the deformation theory of Kodaira-Spencer [KS58] and the algebraic approach of Grothendieck [Gro]. The complex-analytic theory of Teichmüller space was then studied intensely until the 1970s by Lars Ahlfors, Lipman Bers, and others (we refer to [JP13] for a historical introduction).

The story with which this paper is concerned properly started when Weil [Wei58] realized that the Petersson pairing of automorphic forms provides a Hermitian structure on $\mathcal{T}(S)$ now known as the Weil-Petersson metric. Ahlfors [Ahl61] soon proved that this Hermitian metric is Kähler for the natural complex structure on $\mathcal{T}(S)$, unleashing the possibility of a rich symplectic geometry (Weil already claimed this in [Wei58], but in lieu of a proof he wrote that it is "stupid computation").

In the 1970s and 1980s, Teichmüller theory was deeply influenced by the singular geometric vision of William Thurston, and the emphasis partly shifted from complex analysis to hyperbolic geometry. It was eventually understood that the symplectic geometry of Teichmüller space is more intrinsic on the deformation space of hyperbolic structures on S, which we call the Fricke-Klein space $\mathcal{F}(S)$ (and which is in bijection with $\mathcal{T}(S)$ by the Poincaré uniformization theorem).

The symplectic theory of $\mathcal{T}(S)$ and $\mathcal{F}(S)$ was beautifully developed by Wolpert in the 1980s (let us cite [Wol82, Wol83, Wol85]). Wolpert showed that given a (homotopy class of) simple closed curve γ on S, the Hamiltonian flow of the hyperbolic length function $l_\gamma \colon \mathcal{F}(S) \to \mathbb{R}$ is the flow on $\mathcal{F}(S)$ that consists in twisting the hyperbolic structure along γ. Wolpert's work shows that given a pants decomposition of S, the length functions of the pants curves define an integrable Hamiltonian system whose action-angle variables are the famous Fenchel-Nielsen coordinates relative to the pants decomposition.

The full extent of the naturality of the symplectic structure of deformation spaces relative to closed surfaces was established by Goldman [Gol84] (following Atiyah-Bott [AB83]). Goldman showed that there is a natural (complex) algebraic symplectic structure on the character variety $\mathcal{X}(\pi_1(S), G)$ for any (complex) algebraic semisimple Lie group G (where the character variety is the space of (closed) conjugacy classes of group homomorphisms $\pi_1(S) \to G$). A consequence of Goldman's work is that the Fricke-Klein space $\mathcal{F}(S)$, the quasi-Fuchsian space $\mathcal{QF}(S)$, and the deformation space of complex projective structures $\mathcal{CP}(S)$ all enjoy natural symplectic structures, inherited from the character variety for $G = \mathrm{PSL}_2(\mathbb{C})$.

The complex symplectic geometry of $\mathcal{CP}(S)$ was carefully studied by Loustau in [Lou15a] (and [Lou15c]). Let us recall that a complex projective structure on S is a geometric structure defined by an atlas on S whose charts take values in $\mathbb{C}P^1$, and whose transition functions are restrictions of projective linear transformations. Since such an atlas is, in particular, a holomorphic atlas, there is a forgetful projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$. One of the main results of [Lou15a] is that the Schwarzian parametrization of $\mathcal{CP}(S)$ relative to any Lagrangian section of p provides a symplectomorphism $\mathcal{CP}(S) \to \mathrm{T}^*\mathcal{T}(S)$, generalizing a result of Kawai [Kaw96]. Loustau also showed that the symplectic theory of $\mathcal{F}(S)$ developed by Wolpert holomorphically extends to quasi-Fuchsian space, partially extending or clarifying results of Platis [Pla01] and Goldman [Gol04].

The present paper is in part a continuation of [Lou15a] in that we apply general machinery from symplectic geometry to obtain clarifications of results in Teichmüller theory, which up to this point only had proofs relying on idiosyncratic features of the spaces in consideration. Furthermore, we discover new geometric features of these deformation spaces.

One very simple yet key observation is that quasi-Fuchsian space $\mathcal{QF}(S)$ is the complexification of Teichmüller space $\mathcal{T}(S)$. Without precisely defining $\mathcal{QF}(S)$ in this introduction, let us just remember that it is a complex (symplectic) manifold that is isomorphic to $\mathcal{T}(S) \times \overline{\mathcal{T}(S)}$ by the "simultaneous uniformization" biholomorphism $QF\colon \mathcal{T}(S) \times \overline{\mathcal{T}(S)} \to \mathcal{QF}(S)$, whose restriction to the diagonal coincides with the uniformization map $F\colon \mathcal{T}(S) \to \mathcal{F}(S)$. If we recall that $\mathcal{T}(S) \times \overline{\mathcal{T}(S)}$ is the canonical complexification of $\mathcal{T}(S)$, and remember the uniqueness of complexification, we see that simultaneous uniformization is "just" the complexification of uniformization (see Proposition 5.3 for more details). Since $\mathcal{T}(S)$ is a real-analytic Kähler manifold, its complexification $\mathcal{QF}(S)$ must enjoy all the general properties of the complexification of a real-analytic Kähler manifold. In particular, $\mathcal{QF}(S)$ has a natural complex bi-Lagrangian structure as a result of our study in section 3, and we show that the canonical transverse Lagrangian foliations are the well-known foliations of $\mathcal{QF}(S)$ by Bers slices (Theorem 5.5). Other features of this complex bi-Lagrangian structure are not well-known and yet to be explored, such as its holomorphic Riemannian structure, which we largely elucidate in Theorem 5.6, and the hyper-Hermitian structures discussed in section 6. Note that since our construction yields a structure that differs from the Feix-Kaledin hyper-Kähler structure, our bi-Lagrangian and hyper-Hermitian picture of quasi-Fuchsian space is not the same as the hyper-Kähler structure beautifully constructed by Donaldson in [Don03] and further described in [Hod05] and [Tra18].

Let us conclude this introduction by listing the other main applications that we develop in section 5:

- The Weil-Petersson metric can be defined on Fricke-Klein space $\mathcal{F}(S)$ via symplectic geometry, without using the uniformization theorem (§ 5.3).
- The remarkable affine structure in the fibers of the forgetful projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$, which is classically described via the Schwarzian derivative, coincides with the Bott affine structure in the leaves of a Lagrangian foliation (Theorem 5.8).
- The family of affine structures on Teichmüller space provided by the Bers embeddings

coincides with the family of affine structures on a real-analytic Kähler manifold induced by the bi-Lagrangian structure in its complexification (Theorem 5.9).
- The derivative of the Bers embedding at the origin is equal to -1/2 times the musical isomorphism induced by the Weil-Petersson metric (Theorem 5.11).

Outline. Section 1 reviews Lagrangian foliations, affine structures and Bott connections in the real and complex settings. Section 2 discusses real and complex bi-Lagrangian structures and other related notions. Section 3 investigates the complexification of a real-analytic Kähler manifold and its natural complex bi-Lagrangian structure. Section 4 contains an introduction to Teichmüller theory with the necessary background material for the next section. Section 5 applies the general machinery of sections 1, 2, 3 to Teichmüller theory. Section 6 constructs and studies an almost hyper-Hermitian structure in the complexification of a real-analytic Kähler manifold.

A note to the reader: The segment of the paper dealing with general differential geometry (sections 1, 2, 3 and 6) and the segment on Teichmüller theory (sections 4 and 5) are quite dissimilar in flavor and may appeal to different audiences. We tried to make both expositions self-contained while trying to keep a reasonable length; it is inevitable that parts of the paper will feel too detailed and others too condensed to different readers.

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1 Lagrangian foliations and affine structures

In this section, we review basics of Lagrangian foliations, define affine structures and discuss Bott connections, both in the real (smooth) setting and in the complex setting. Good references for Lagrangian foliations and Bott connections in the real setting include [Wei71], [Vai89] and [FY13].

1.1 Lagrangian foliations

Let M be a smooth manifold. A *symplectic structure* on M is a differential 2-form $\omega \in \Omega^2(M, \mathbb{R})$ which is closed (such that $\mathrm{d}\omega = 0$) and nondegenerate (as a bilinear form on $\mathrm{T}M$). A *Lagrangian* *submanifold* is a smoothly embedded submanifold $f : N \to M$ which is isotropic (*i.e.* such that $f^*\omega = 0$) and of maximal dimension among isotropic submanifolds (*i.e.* $\dim N = \dim M/2$). A *Lagrangian foliation* of M is a foliation $\mathcal{F}$ of M such that every leaf $N \in \mathcal{F}$ is Lagrangian.

If M is a complex manifold, a *complex symplectic structure* ω is a closed nondegenerate $(2,0)$-form. The assumption on ω implies that it is a holomorphic form: it is locally of the form $\omega = \omega_{ij}\, \mathrm{d}z^i \wedge \mathrm{d}z^j$ in a complex chart (z^i) on M, where the ω_{ij}s are holomorphic functions on M. Note that we use the Einstein notation that implies summation over repeated indices. In this category, a *complex Lagrangian submanifold* is a holomorphically embedded half-dimensional submanifold $f : N \to M$ such that $f^*\omega = 0$. Note that both the real and imaginary parts of ω are real symplectic structures on M, and a complex Lagrangian submanifold is real Lagrangian for both $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$. A *complex Lagrangian foliation* is a foliation $\mathcal{F}$ of M such that each leaf $N \in \mathcal{F}$ is a complex Lagrangian submanifold.

In either the smooth or complex setting, let $p : M \to B$ be a fiber bundle where (M, ω) is a symplectic manifold. The fiber bundle $p : M \to B$ is called a *Lagrangian fibration* if for every $b \in B$, the fiber $p^{-1}(b)$ is Lagrangian in M. Therefore, the total space of any Lagrangian fibration has a Lagrangian foliation. Conversely, given a symplectic manifold with a Lagrangian foliation, projection to the leaf space yields a Lagrangian fibration in any sufficiently small open set.

Example 1.1 (Cotangent bundle). A fundamental example of a symplectic manifold and a Lagrangian fibration is provided by the cotangent bundle of any manifold. Let N be a manifold either in the smooth or complex category. Let $M = \mathrm{T}^*N$ denote the total space of the cotangent bundle $p : \mathrm{T}^*N \to N$. Note that in the complex setting, we let T^*N denote the holomorphic cotangent bundle *i.e.* the complex dual of the holomorphic tangent bundle $\mathrm{T}^{(1,0)}N$. Then M admits a *canonical* 1*-form* ξ and a *canonical symplectic structure* ω. The canonical form ξ is defined at a point $\alpha \in M$ as the pull-back $\xi_\alpha = p^*\alpha$ of the covector $\alpha \in \mathrm{T}^*M$, and the canonical symplectic structure is $\omega = \mathrm{d}\xi$. The expression of ξ and ω in coordinates is as follows. Choosing local coordinates $(q^1, \ldots, q^n)$ on N defines natural coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on $M = \mathrm{T}^*N$: the point with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is the covector $p_i\, \mathrm{d}q^i \in \mathrm{T}^*N$. In these coordinates, the canonical 1-form and symplectic structure are given by $\xi = p_i\, \mathrm{d}q^i$ and $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$ [1]. It is straightforward to check that the bundle projection $p : \mathrm{T}^*N \to N$ is a Lagrangian fibration for the canonical symplectic structure.

Lagrangian foliations satisfy the following structure theorems (both in the smooth and complex settings):

Theorem 1.2 (Theorem of Darboux-Lie)**.** *Let (M, ω) be a symplectic manifold with a Lagrangian foliation $\mathcal{F}$. There exists local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on any sufficiently small open set U such that:*

(i) $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ are Darboux coordinates for the symplectic structure up to sign:

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i \ .$$

(ii) The leaves of the foliations are the level sets of $q = (q_1, \ldots, q_n)$.

The Darboux-Lie theorem implies that, given a Lagrangian submanifold $f : N \to M$ which is transverse to a Lagrangian foliation $\mathcal{F}$ of M, there is a canonical identification between a neighborhood of the zero section in T^*N and a neighborhood of $f(N) \subset M$.

Theorem 1.3 (Weinstein's symplectic tubular neighborhood theorem [Wei71])**.** *Let (M, ω) be a symplectic manifold with a Lagrangian foliation $\mathcal{F}$ and let $f : N \to M$ be a Lagrangian submanifold transverse to $\mathcal{F}$. Denote by $s_0 : N \to \mathrm{T}^*N$ the zero section of the cotangent bundle. Then there exists a unique germ of a diffeomorphism $\phi : U \to V$ where U is a neighborhood of $f(N)$ in M and V is a neighborhood of $s_0(N)$ in T^*N such that:*

[1] Some authors' definition of the canonical symplectic structure ω differs from ours by a minus sign.

(i) $\phi \circ f = s_0$.
(ii) ϕ *is a symplectomorphism.*
(iii) ϕ *is fiber-preserving.*

1.2 Affine structures

Let M be a smooth or complex manifold of dimension n.

Definition 1.4. An *affine structure* on M is equivalently the data of:
(i) A compatible (X, G)-structure on M, where $X = \mathbb{A}^n$ is the standard n-dimensional affine space and $G = \mathrm{Aff}(\mathbb{A}^n)$ is the group of affine transformations of $\mathbb{A}^n$.
(ii) A flat torsion-free [complex] connection ∇ in the tangent bundle $\mathrm{T}M$.

We pause to quickly review the terminology appearing in the previous definition:
- The standard n-dimensional affine space is $\mathbb{A}^n = k^n$ and its group of affine transformations is $G = \mathrm{Aff}(k^n) \approx \mathrm{GL}_n(k) \ltimes k^n$, where $k = \mathbb{R}$ or $\mathbb{C}$.
- A *compatible* (X, G)*-structure* on M is given by an atlas of smooth/complex charts with values in X such that the transition functions coincide with the action of elements of G on X. For more background on (X, G)-structures, see § 4.1.1.
- In the complex setting, we require the connection ∇ to be a *complex connection*, meaning that the almost complex structure I on M is parallel: $\nabla I = 0$. Any flat torsion-free complex connection is in fact a *holomorphic connection*, meaning that $\nabla_X Y$ is a holomorphic vector field whenever X and Y are holomorphic vector fields.

The fact that the two definitions of an affine structure given above are equivalent is rather elementary and left to the reader. A third possible characterization exists in terms of a locally defined, free and transitive action of the vector space $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) on M, but we will not discuss this approach here.

An (X, G)-structure is called *complete* when it is given as the quotient of the model space X by a discrete subgroup of G (in the situation where X is simply connected), see § 4.1.1 for details. In the case of affine structures, this is equivalent to saying that the connection ∇ is geodesically complete.

1.3 Bott connection

1.3.1 Real Bott connection

Let us first review Bott connections in the real setting: in what follows, M is a smooth manifold.

We begin with the notion of a partial connection in a vector bundle along a distribution:

Definition 1.5. Let $V \to M$ be a vector bundle on M and $L \subset \mathrm{T}M$ a distribution on M (*i.e.* a smooth subbundle of $\mathrm{T}M$). A *partial (linear) connection* in V along L is a map

$$\begin{aligned} \nabla : \Gamma(L) \times \Gamma(V) &\to \Gamma(V) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

which is $C^\infty(M, \mathbb{R})$-linear in X and a derivation in s, *i.e* satisfies the Leibniz rule $\nabla_X(fs) = X(f)s + f\nabla_X s$ for any function $f \in C^\infty(M, \mathbb{R})$. We use the notation Γ above to denote the space of smooth sections of a vector bundle.

One can then define the Bott connection in the annihilator of an involutive distribution as follows. Let L be an involutive[2] distribution on M and $L^\perp \subset \mathrm{T}^*M$ be the annihilator of L, *i.e.* the subbundle of T^*M defined by $L_x^\perp := \{\alpha \in T_x^*M \colon L_x \subseteq \ker \alpha\}$.

Definition 1.6. The *Bott connection* in $L^\perp$ along L is the partial connection defined by $\nabla_X \alpha = \mathcal{L}_X \alpha$, where $\mathcal{L}$ is the Lie derivative.

It is a straightforward exercise to check that this defines a partial connection, and that moreover this connection is flat:

Proposition 1.7. *The Bott connection in $L^\perp$ along L is flat.*

Now let ω be a symplectic structure on M. Consider a Lagrangian involutive distribution L, or equivalently a Lagrangian foliation $\mathcal{F}$ of M (L is the tangent distribution to the foliation $\mathcal{F}$).

Definition 1.8. The *Bott connection in L* is the partial connection in L along L obtained from the Bott connection in $L^\perp$ along L using the isomorphism $L \approx L^\perp$ induced by the symplectic musical isomorphism $\flat : \mathrm{T}M \to \mathrm{T}^*M$.

Recall that the musical isomorphism $\flat : \mathrm{T}M \to \mathrm{T}^*M$, $u \mapsto u^\flat$ is defined by $u^\flat = \omega(u, \cdot)$. It restricts to an isomorphism $L \to L^\perp$.

Here are essential facts about the Bott connection in a Lagrangian foliation:

Proposition 1.9. *Let (M, ω) be a real symplectic manifold, let $\mathcal{F}$ be a Lagrangian foliation of M and denote by ∇ the Bott connection as above.*

(i) ∇ is characterized by the formula

$$X \cdot \omega(Y, Z) = \omega(\nabla_X Y, Z) + \omega(Y, [X, Z])$$

where X and Y are vector fields on M tangent to the foliation and Z is any vector field.

(ii) ∇ is flat and torsion-free.

Note that the Bott connection may extend to a "full" connection in $\mathrm{T}M$, but there is no canonical way to choose such an extension in general. In the next section, we will see that given two transverse Lagrangian foliations, there exists a unique symplectic connection in M that extends both Bott connections (Proposition 2.2). On the other hand, the Bott connection restricts to a full connection in any leaf of the foliation, which moreover is flat and torsion-free by Proposition 1.9. It thus defines an affine structure in any leaf of the foliation according to Definition 1.4:

Theorem 1.10. *The Bott connection equips every leaf of a Lagrangian foliation with a natural affine structure.*

We shall call this affine structure in the leaves of a Lagrangian foliation the *Bott affine structure*.

Theorem 1.10 is closely related to the Liouville-Arnold theorem in the following way. Recall that a Lagrangian foliation is locally a Lagrangian fibration. Choosing coordinates on the base of a Lagrangian fibration defines an integrable Hamiltonian system on the total space. In this situation, the Liouville-Arnold theorem guarantees the existence of "action-angle coordinates" that are canonical for the foliation, as in the Darboux-Lie Theorem 1.2, and such that there is an $\mathbb{R}^n$-action (defining the angle coordinates) in the level sets of the action coordinates. These level sets are exactly the leaves of the foliation. This (local) $\mathbb{R}^n$-action is another way to think about the affine structure of Theorem 1.10.

[2] Recall that a smooth distribution $L \subset \mathrm{T}M$ is called *involutive* (or *integrable*) if the space of sections of L is stable under the Lie bracket. The Frobenius theorem states that a distribution is involutive if and only if it is the tangent space to a smooth foliation.

1.3.2 Complex Bott connection

Let us now turn to the Bott connection in the complex setting. In what follows, M is a complex manifold. Assume that M is equipped with a complex symplectic structure ω and that $\mathcal{F}$ is a complex Lagrangian foliation of M. Denote by ω_1 and ω_2 the real and imaginary parts of ω so that $\omega = \omega_1 + i\omega_2$. These are real symplectic structures on M, and $\mathcal{F}$ is a real Lagrangian foliation with respect to both ω_1 and ω_2.

The *complex Bott connection* in the distribution $L = \mathrm{T}\mathcal{F}$ is characterized as follows:

Theorem 1.11. *There is a unique partial connection ∇ in L along L such that:*

(i) ∇ is a flat and torsion-free partial complex connection.

(ii) $\nabla = \nabla^1 = \nabla^2$, where ∇^i is the Bott connection of ω_i in L ($i = 1, 2$).

The proof of this theorem easily follows from the properties of real Bott connections stated in Proposition 1.9. One derives from Theorem 1.11 the following fundamental property of complex Lagrangian foliations:

Theorem 1.12. *The complex Bott connection equips every leaf of a complex Lagrangian foliation with a natural complex affine structure.*

1.3.3 Fundamental example: cotangent bundles

Let us work either in the smooth or complex setting in what follows. Recall from Example 1.1 that if $M = \mathrm{T}^*N$ is the total space of a cotangent bundle, then M carries a canonical symplectic structure ω and the bundle projection $p : M \to N$ is a Lagrangian fibration. By Theorem 1.10 or Theorem 1.12, each fiber is equipped with a natural affine structure, given by the Bott connection. On the other hand, any fiber has an "obvious" affine structure since it is a vector space. It turns out that these two affine structures coincide:

Theorem 1.13. *The Bott affine structure in any fiber of T^*N is the same as its affine structure as a vector space.*

Since this theorem is key to proving Theorem 5.8, let us produce a complete proof, though it is a straightforward computation.

Proof. Let us write a proof in the real setting, the extension to the complex setting is an immediate consequence of Theorem 1.11. The Bott connection ∇ is characterized by the identity

$$\omega(\nabla_X Y, Z) = X \cdot \omega(Y, Z) - \omega(Y, [X, Z])$$

for any vector fields X and Y on $M = \mathrm{T}^*N$ that are tangent to the foliation and for any vector field Z on M (cf. Proposition 1.9). Let us work on some open set $\mathrm{T}^*U \subset M$ with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ as in Example 1.1. The canonical symplectic structure ω is given by

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$$

and the vector fields X, Y, and Z can be written as follows (recall that we use the Einstein notation for writing tensors):

$$X = a_i \frac{\partial}{\partial p_i} \qquad Y = b_i \frac{\partial}{\partial p_i} \qquad Z = c^i \frac{\partial}{\partial q^i} + d_i \frac{\partial}{\partial p_i} \,.$$

It is now a direct computation that:

$$X \cdot \omega(Y, Z) - \omega(Y, [X, Z]) = a_i \, c^j \, \frac{\partial b_j}{\partial p_i} \,. \qquad (1)$$

On the other hand, writing $\nabla_X Y$ in coordinates as $\nabla_X Y = v_j \dfrac{\partial}{\partial p_j}$ gives

$$\omega(\nabla_X Y, Z) = v_j\, c^j \ . \tag{2}$$

Equating (1) and (2) yields $v_j = a_i \dfrac{\partial b_j}{\partial y^i}$, which gives us the expression of $\nabla_X Y$ in these coordinates:

$$\nabla_X Y = a_i \frac{\partial b_j}{\partial p_i} \frac{\partial}{\partial p_j} \ .$$

Conclude by observing that this is the expression of the standard covariant derivative in $\mathbb{R}^n$ in the coordinates $(p_1, \dots p_n)$. □

2 Bi-Lagrangian structures

In this section, we review bi-Lagrangian and para-Kähler structures in the real setting and study properties of the bi-Lagrangian connection. Then, we turn our attention to complex bi-Lagrangian structures. A good reference for bi-Lagrangian structures in the real setting is [EST06].

2.1 Real bi-Lagrangian structures

In what follows, M is a smooth manifold.

Definition 2.1. A *bi-Lagrangian structure* in M is the data of a symplectic structure ω and an ordered pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transverse Lagrangian foliations.

Bi-Lagrangian structures have also been called *bipolarizations*, as a Lagrangian foliation is sometimes called a (real) *polarization*, especially in mathematical physics. We shall soon see that a bi-Lagrangian structure is also the same as a *para-Kähler structure* (cf § 2.2).

A connection ∇ in a symplectic manifold M is called a *symplectic connection* if it is torsion-free and parallelizes the symplectic structure: $\nabla\omega = 0$. Symplectic connections always exist but they are not unique, contrary to Riemannian connections. However, the additional data of a bi-Lagrangian structure is enough to determine a unique natural connection ∇ called the *bi-Lagrangian connection* which can be characterized as follows.

Proposition 2.2. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. There exists a unique symplectic connection ∇ on M which extends both the Bott connections in $\mathcal{F}_1$ and in $\mathcal{F}_2$.*

One can show that the bi-Lagrangian connection ∇ is in fact the unique symplectic connection in M which satisfies the apparently weaker condition that it preserves both foliations, see Theorem 2.6.

2.2 Para-complex and para-Kähler structures

The algebra of *para-complex numbers*[3] is the commutative algebra

$$\mathbb{R}[X]/(X^2 - 1) = \{a + fb : (a,b) \in \mathbb{R}^2, f^2 = 1\} \ .$$

[3]Para-complex numbers are also sometimes called *split-complex numbers*, *hyperbolic numbers* and a variety of other names that are listed in [Wik17].

Para-complex structures on manifolds are the para-complex analog of complex structures. We refer to [CFG96] for a survey of para-complex geometry.

Definition 2.3. Let M be a smooth manifold. An *almost para-complex structure* on M is a smooth field of endomorphisms $F \in \mathrm{End}(\mathrm{T}M)$ such that $F^2 = \mathbf{1}$ and $\mathrm{tr}(F) = 0$.

Call L_1 and L_2 the $+1$ and -1 eigendistributions of an almost para-complex structure F, so the tangent bundle of M splits as $\mathrm{T}M = L_1 \oplus L_2$. The condition $\mathrm{tr}(F) = 0$ amounts to saying that $\dim L_1 = \dim L_2$[4]. The almost para-complex structure F is called *integrable* if L_1 and L_2 are involutive, in other words there are two transverse foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of M such that L_i is the tangent subbundle to $\mathcal{F}_i$ ($i \in \{1,2\}$). When F is integrable, it is also just called a *para-complex structure*. It is clear that any ordered pair of equidimensional transverse distributions (L_1, L_2) conversely determines a unique almost para-complex structure F on M as above. Therefore, the data of a para-complex structure on M is equivalent to the data an ordered pair of equidimensional transverse foliations.

Next we define para-Kähler structures, which are the para-complex analog of Kähler structures (compare with Definition 3.7).

Definition 2.4. A *para-Kähler* structure in a smooth manifold M is the data of (g, F, ω), where:
- g is a pseudo-Riemannian metric in M,
- F is an (almost) para-complex structure in M and
- ω is an (almost) symplectic structure in M

such that:
(i) $\omega(u, v) = g(Fu, v)$ for any u, v (*compatibility condition*)
(ii) F is parallel for the Levi-Civita connection of g: $\nabla F = 0$ (*integrability condition*).

It is easy to see from this definition that the signature of g must be (n, n) where $\dim M = 2n$, such a pseudo-Riemannian metric is called a *neutral metric*. As in the complex case, the integrability condition is equivalent to the simultaneous integrability of F as a para-complex structure and closedness of ω as a 2-form, so that a para-Kähler manifold is a para-complex manifold and a symplectic manifold in addition to a pseudo-Riemannian manifold, and these three structures are mutually compatible.

If (g, F, ω) is a para-Kähler structure on M, then one quickly checks that the eigendistributions L_1 and L_2 of F are isotropic for g and ω. In particular, the two transverse foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ defined by F are Lagrangian for ω and thus define a bi-Lagrangian structure on M. Conversely, it is clear that any bi-Lagrangian structure on M determines a unique para-Kähler structure (g, F, ω).

Proposition 2.5. *Let M be a smooth manifold. There is a 1-1 correspondence between bi-Lagrangian structures $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ on M and para-Kähler structures (g, F, ω) on M, where the tangent distributions to $\mathcal{F}_1$ and $\mathcal{F}_2$ are respectively the $+1$ and -1 eigendistributions of the para-complex structure F.*

2.3 Properties of the bi-Lagrangian connection

Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. We shall call the neutral metric g of the associated para-Kähler structure the *bi-Lagrangian metric*. Recall that there is a unique symplectic connection in M extending both the Bott connections in $\mathcal{F}_1$ and $\mathcal{F}_2$ called the *bi-Lagrangian connection* (cf Proposition 2.2). The following theorem ensures that the bi-Lagrangian connection can be alternatively defined as the Levi-Civita connection of g.

[4] When this condition is not required, F is called an *almost product structure*.

Theorem 2.6. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold and denote by (g, F, ω) the associated para-Kähler structure. The bi-Lagrangian connection of M is the unique torsion-free connection ∇ satisfying the following equivalent conditions:*

(i) ∇ parallelizes g: $\nabla g = 0$.

(ii) ∇ parallelizes ω and F: $\nabla \omega = 0$ and $\nabla F = 0$.

(iii) ∇ parallelizes ω and preserves both foliations $\mathcal{F}_1$ and $\mathcal{F}_2$.

Knowing that the bi-Lagrangian connection ∇ is the Levi-Civita connection of the bi-Lagrangian metric g allows one to compute it explicitly as follows. Let ∇^{B_i} denote the Bott connection in $\mathcal{F}_i$ ($i \in \{1, 2\}$). Recall that for any vector $u \in \mathrm{T}M$, we write $u = u_1 + u_2$ where u_i is tangent to the foliation $\mathcal{F}_i$ ($i \in \{1, 2\}$). Then for any vector fields X and Y:

$$\nabla_X Y = \nabla^{B_1}_{X_1} Y_1 + \nabla^{B_2}_{X_2} Y_2 + [X_1, Y_2]_2 + [X_2, Y_1]_1 \ . \tag{3}$$

In particular one can derive from (3) that both foliations are totally geodesic. Note that they are also flat since the Bott connections ∇^{B_1} and ∇^{B_2} are flat there. Next we compute the curvature tensor R of the bi-Lagrangian connection, which is straightforward using (3):

Proposition 2.7. *Let X, Y, and Z be vector fields on M.*

- *If $X = X_i$ and $Y = Y_i$ are both tangent to $\mathcal{F}_i$ where $i \in \{1, 2\}$, then $R(X, Y) = 0$.*
- *If $X = X_1$ is tangent to $\mathcal{F}_1$ and $Y = Y_2$ is tangent to $\mathcal{F}_2$, then:*

$$\begin{aligned} R(X,Y)Z = {} & \nabla_X [Y, Z_1]_1 - [[X,Y]_2, Z_1]_1 - \nabla_{[X,Y]_1} Z_1 - [Y, \nabla_X Z_1]_1 \\ & - \nabla_Y [X, Z_2]_2 - [[X,Y]_1, Z_2]_2 - \nabla_{[X,Y]_2} Z_2 - [X, \nabla_Y Z_2]_2 \ . \end{aligned}$$

By linearity and antisymmetry of the curvature tensor R in X and Y, Proposition 2.7 determines R completely. Note that $R(X, Y)$ always preserves both distributions in the sense that $R(X, Y)Z$ is tangent to $\mathcal{F}_i$ whenever Z is tangent to $\mathcal{F}_i$. Also observe that while we established that the restriction of ∇ to a leaf of either foliations is flat, the fact that $R(X, Y) = 0$ whenever X and Y are tangent to one foliation is somewhat stronger. It implies that parallel transport with respect to ∇ of any tensor on M along a path contained in a leaf is independent of the path. We shall make use of that property in the construction of Theorem 6.10, so let us record this precisely:

Corollary 2.8. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold and denote by ∇ the bi-Lagrangian connection. Let $\gamma : [0, 1] \to M$ be a smooth path contained in a leaf of either foliations $\mathcal{F}_i$ and let T be a tensor on M at $\gamma(0)$. Then the parallel transport of T along γ with respect to ∇ does not depend on the choice of γ in its homotopy class in the leaf rel. endpoints.*

The proof of Corollary 2.8 can be adapted effortlessly from a standard proof that the vanishing of the curvature of a connection is equivalent to the local path-independence of parallel transport, see e.g. [Vor09, Theorem 11.1].

2.4 Complex bi-Lagrangian structures

We now turn to complex bi-Lagrangian structures, which are a natural extension of real bi-Lagrangian structures in the complex setting.

In what follows, M is a complex manifold. We let J denotes the (integrable) almost complex structure of M. We recall that an almost complex structure on a smooth manifold M is a tensor field $J \in \mathrm{End}(\mathrm{T}M)$ such that $J^2 = -\mathbf{1}$. There is a notion of integrability for almost complex structures in terms of the vanishing of their so-called *Nijenhuis tensor*; the Newlander-Nirenberg theorem says that J is integrable if and only if M can be given the structure of a complex manifold such that J is the natural almost complex structure on M, which corresponds to scalar multiplication by i in each tangent space.

Definition 2.9. A *complex bi-Lagrangian structure* in M is the data of a complex symplectic structure ω and an ordered pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transverse complex Lagrangian foliations.

Similarly to the real case, the data of a complex bi-Lagrangian structure in M is equivalent to the data of a *bicomplex para-Kähler structure* (g, F, ω), but before we define such structures, we need to review basic facts about holomorphic metrics.

2.5 Holomorphic metrics

By definition, a *holomorphic (Riemannian) metric* g on M is a holomorphic symmetric complex bilinear form on M (*i.e.* a holomorphic section of the symmetric product $S^2\,\mathrm{T}^*M$) which is nondegenerate. The following proposition is elementary.

Proposition 2.10. *Let g be a holomorphic Riemannian metric on M, denote by g_1 and g_2 its real and imaginary parts respectively. Then*

(i) g_1 and g_2 are neutral metrics on M.

(ii) $g_1(u,v) = g_2(Ju,v)$ for any tangent vectors u, v at a same point of M.

We recall that a neutral metric on M is a pseudo-Riemannian metric of signature (n,n), where $\dim_{\mathbb{R}} M = 2n$.

The "fundamental theorem of Riemannian geometry" holds for holomorphic metrics, more precisely:

Theorem 2.11. *Let g be a holomorphic metric on M. There exists a unique torsion-free holomorphic connection ∇ which parallelizes g, called its* holomorphic Levi-Civita connection. *Moreover, ∇ is the Levi-Civita connection of both the real and imaginary parts of g.*

The proof of Theorem 2.11 is the same as the usual proof for Riemannian metrics: the fact that ∇ is torsion-free and parallelizes g implies that it has to satisfy the *Koszul formula*:

$$\begin{aligned} 2\,g(\nabla_X Y, Z) &= X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y) \end{aligned}$$

which shows existence and uniqueness. The fact that ∇ is the Levi-Civita connection of g_1 and g_2 is derived by taking the real and imaginary parts of the formula. Since $g_1(\cdot,\cdot) = g_2(J\cdot,\cdot)$, the fact that g_1 and g_2 are parallel implies that J is parallel, so that ∇ is a complex connection. Finally, we can also derive directly from the Koszul formula and g being a holomorphic tensor field that ∇ is a holomorphic connection.

2.6 Bicomplex Kähler structures

The algebra of *bicomplex numbers* $\mathbb{BC}$ is the unital associative algebra over the real numbers generated by three elements i, j, and f satisfying the *bicomplex relations*:

$$\begin{gathered} i^2 = -1 \quad j^2 = -1 \quad f^2 = +1 \\ ij = ji = f\ . \end{gathered}$$

The algebra of bicomplex numbers $\mathbb{BC}$ is a 4-dimensional algebra over $\mathbb{R}$: a generic bicomplex number is written $q = a + ib + jc + kd$ with $(a,b,c,d) \in \mathbb{R}^4$. One quickly sees that $\mathbb{BC}$ can be simply be described as $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ by writing $q = (a+bi) + (c+di)j$. We refer to [LESSV15] for the reader interested to learn more about bicomplex numbers.

Definition 2.12. An *bicomplex structure* on a smooth manifold M is the data of an ordered triple (I, J, K) where:

- I and J are integrable almost complex structures.
- F is an integrable para-complex structure.
- I, J, F satisfy the bicomplex relations as above.

Of course, the data of only two of the three structures I, J, and F is enough to determine the third one via the relation $IJ = F$. This allows us to give the following equivalent definitions of a *holomorphic* bicomplex structure on M when the complex structure on M corresponding to J is already given:

Definition 2.13. Let (M, J) be a complex manifold. A *holomorphic bicomplex structure* on M is equivalently the data of:

(i) An integrable almost complex structure I which is complex linear as an endomorphism of $\mathrm{T}M$ and holomorphic as a tensor field on M, and such that $\mathrm{tr}(IJ) = 0$.

(ii) An integrable almost para-complex structure F which is complex linear as an endomorphism of $\mathrm{T}M$ and holomorphic as a tensor field on M.

(iii) An ordered pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transverse holomorphic foliations of M by half-dimensional complex submanifolds.

This definition of a holomorphic bicomplex structure is seemingly weaker than another natural definition, namely the data of an atlas on M with bicomplex-holomorphic transition functions (as in e.g. [BW11]), but they turn out to be equivalent. This follows from the fact that a holomorphic bicomplex manifold is locally biholomorphic to a product of complex manifolds, which is the complex declination of the well-known fact that an integrable para-complex structure induces a local product structure (see e.g. [Lib54] for details). A discussion and examples of bicomplex manifolds in terms of bicomplex atlases can be found in [BW11].

We are now ready to define bicomplex Kähler structures:

Definition 2.14. Let M be a complex manifold. A *(holomorphic) bicomplex Kähler structure* on M is the data of (g, F, ω), where:

- g is a holomorphic metric in M,
- F is a para-complex structure in M defining a (holomorphic) bicomplex structure, and
- ω is a complex symplectic structure in M

such that:

(i) $\omega(u, v) = g(Fu, v)$ for any u, v (*compatibility condition*)

(ii) F is parallel for the Levi-Civita connection of g: $\nabla F = 0$ (*integrability condition*).

As in the real setting, it is straightforward to show the following proposition.

Proposition 2.15. *Let M be a complex manifold. There is a 1-1 correspondence between complex bi-Lagrangian structures $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ on M and holomorphic bicomplex Kähler structures (g, F, ω) on M, where the tangent distributions to $\mathcal{F}_1$ and $\mathcal{F}_2$ are respectively the $+1$ and -1 eigendistributions of the para-complex structure F.*

2.7 Complex bi-Lagrangian metric and connection

Let M be a complex manifold equipped with a complex bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ and denote by (g, F, ω) the associated holomorphic bicomplex Kähler structure (cf Proposition 2.15). We shall call g the *complex bi-Lagrangian metric* and its Levi-Civita connection ∇ the *complex bi-Lagrangian connection*.

Theorem 2.16. *The complex bi-Lagrangian connection ∇ is the unique torsion-free holomorphic connection in M which satisfies the equivalent conditions:*

(i) ∇ *parallelizes* g.

(ii) ∇ *parallelizes* ω *and preserves both foliations* $\mathcal{F}_1$ *and* $\mathcal{F}_2$.

(iii) ∇ *parallelizes* g *and the bicomplex structure* (I, J, F).

Moreover, ∇ *extends both the complex Bott connections in* $\mathcal{F}_1$ *and in* $\mathcal{F}_2$.

Let us write ω in terms of its real and imaginary parts: $\omega = \omega_1 + i\omega_2$. Note that $(\omega_1, \mathcal{F}_1, \mathcal{F}_2)$ and $(\omega_2, \mathcal{F}_1, \mathcal{F}_2)$ are both real bi-Lagrangian structures in M, let us call them the real and imaginary parts of the complex bi-Lagrangian structure.

Theorem 2.17. *The bi-Lagrangian metrics associated to the real and imaginary parts of the complex bi-Lagrangian structure are respectively the real and imaginary parts of the complex bi-Lagrangian metric. Moreover, the complex bi-Lagrangian connection is equal to the real bi-Lagrangian connection for both the real and imaginary parts of the complex bi-Lagrangian structure.*

The proofs of Theorem 2.16 and Theorem 2.17 are easily reduced to a combination of Theorem 1.11, Theorem 2.6, Theorem 2.11, and Proposition 2.15.

3 Bi-Lagrangian structure in the complexification of a Kähler manifold

In this section, we show that the complexification of a real-analytic Kähler manifold enjoys a natural complex bi-Lagrangian structure and study some of its properties.

3.1 Complexification of real-analytic manifolds

Let us recall the essentials of complexification of real-analytic manifolds.

Definition 3.1. Let M be a complex manifold and denote by J its almost complex structure. A real-analytically embedded submanifold $f : N \to M$ is called *maximal totally real* if for all $p \in N$:

$$\mathrm{T}_p N \oplus J(\mathrm{T}_p N) = \mathrm{T}_p M \ .$$

In this case the embedding $f : N \to M$ is called a *complexification of* N.

One can characterize complexifications of real-analytic manifolds in terms of fixed points of anti-holomorphic involutions:

Proposition 3.2. *Let N be a real-analytic submanifold of a complex manifold M. Then M is a complexification of N if and only if there exists an anti-holomorphic involution $\tau : U \to U$, where U is a neighborhood of N in M, such that N is the set of fixed points of τ.*

The following theorem guarantees existence of a complexification of any real-analytic manifold and uniqueness "up to restriction" (so that the *germ* of a complexification is unique):

Theorem 3.3. *Let N be a real-analytic manifold.*

(i) There exists a complexification $f : N \to M$.

(ii) Let $f_1 : N \to M_1$ and $f_2 : N \to M_2$ be two complexifications. There exists a unique germ of a biholomorphism

$$\phi : U_1 \to U_2$$

where U_i is a connected neighborhood of $f_i(N)$ in M_i ($i = 1, 2$), such that $f_2 = \phi \circ f_1$.

A fundamental fact about complexification of a real-analytic manifold is that any analytic tensor field extends uniquely to (a germ of) a holomorphic tensor field in the complexification:

Proposition 3.4. *Let N be a real-analytic manifold and let $f : N \to M$ be a complexification. Let T be a real-analytic tensor field on N, then there exists a unique germ of a holomorphic tensor field T^c in a neighborhood of $f(N)$ in M such that $f^*T^c = T$.*

The proof of this proposition, in local coordinates, boils down to standard analytic continuation using power series. Let us clarify that by *holomorphic tensor field*, we mean a holomorphic section of the tensor product of a finite number of copies of the holomorphic tangent bundle and its dual. Here are a few examples of this phenomenon:

- Any real-analytic function on N locally extends to a holomorphic function on M.
- Any real-analytic symplectic structure on N locally extends to a complex symplectic structure on M.
- Any real-analytic Riemannian metric on N locally extends to a holomorphic metric on M.

3.2 Complexification of complex manifolds

Given a real-analytic manifold N, even though there exists an essentially unique complexification M, there is no canonical model for M in general. However, if N happens to be a complex manifold then such a canonical complexification exists:

Proposition 3.5. *Let N be a complex manifold, denote by I_0 its almost complex structure. Let $N^c := N \times N$; equip N^c with the integrable almost complex structure $J := I_0 \oplus -I_0$ (in other words, $N^c = N \times \overline{N}$ according to a standard notation). Then the diagonal embedding $f : N \to N^c$ is a complexification of N.*

Let us call $N^c := N \times \overline{N}$ the *canonical complexification* of N.

Let $(z^1, \ldots z^n)$ be a system of local holomorphic coordinates in N, denote $w^i = \overline{z^i}$ the conjugates in a copy of N. Then $(z^1, \ldots, z^n, w^1, \ldots, w^n)$ is a system of local holomorphic coordinates in N^c. The complexification map $f : N \to N^c$ is given in these coordinates by $f(z^1, \ldots, z^n) = (z^1, \ldots, z^n, \overline{z^1}, \ldots, \overline{z^n})$.

Note that in this situation, the anti-holomorphic involution τ of Proposition 3.2 is simply the map $\tau(x, y) = (y, x)$, defined everywhere in $N^c = N \times \overline{N}$. In the coordinates (z, w) above, it is given by $\tau(z, w) = (\overline{w}, \overline{z})$.

Remarkably, when N is a complex manifold, any complexification M admits two natural transverse holomorphic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ by half-dimensional complex submanifolds (at least in a neighborhood of N in M). When $M = N^c$ is given as the canonical complexification, these foliations are simply the vertical and horizontal foliations of the product $N^c = N \times \overline{N}$. In the coordinates $(z^1, \ldots, z^n, w^1, \ldots, w^n)$ above, the vertical foliation is given by $\{z = z_0\}$ where $z_0 \in \mathbb{C}^n$ is a constant, similarly the horizontal foliation is given by $\{w = w_0\}$. Let us record this in a definition:

Definition 3.6. Let N be a complex manifold and let M be a complexification. The ordered pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transverse holomorphic foliations of (a neighborhood of N in) M described above will be called the *canonical pair of foliations* of M.

Note that these foliations are not defined in any complexification of N when N is not equipped with a complex structure to begin with, in other words they depend on the complex structure on N.

3.3 Complexification of Kähler manifolds

We recall the definition of a Kähler manifold in order to fix notations and conventions:

Definition 3.7. A *Kähler* structure on a smooth manifold N is the data of (g, I, ω), where:

- g is a Riemannian metric in N,
- I is an (integrable) almost complex structure, and
- ω is an (almost) symplectic structure in N

such that:

(i) $\omega(u, v) = g(Iu, v)$ for any tangent vectors u, v (*compatibility condition*)

(ii) I is parallel for the Levi-Civita connection of g: $\nabla I = 0$ (*integrability condition*).

The integrability condition is equivalent to the simultaneous integrability of I as a complex structure and closedness of ω as a 2-form, so that a Kähler manifold is a complex manifold and a symplectic manifold in addition to a Riemannian manifold, and these three structures are mutually compatible.

Now let (N, g_0, I_0, ω_0) be a *real-analytic Kähler manifold*, *i.e.* a Kähler manifold such that g_0 (and I_0 and ω_0, automatically) are real-analytic tensor fields for the real-analytic structure underlying the complex-analytic structure. Let $N \hookrightarrow M$ be a complexification of N (we can take the canonical complexification of Proposition 3.5). Denote by J the (integrable) almost complex structure on M. By Proposition 3.4, in a neighborhood U of N in M, the tensor fields g_0, I_0, and ω_0 admit unique holomorphic extensions, namely:

- The Riemannian metric g_0 extends to a holomorphic metric g_0^c (see § 2.5 for the definition of a holomorphic metric).
- The almost complex structure I_0 extends holomorphically to a complex endomorphism I_0^c of $\mathrm{T}M$ which squares to -1.
- The symplectic form ω_0 extends to a complex symplectic form ω_0^c.

In addition, M enjoys two canonical transverse holomorphic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ as we saw in Definition 3.6. The following theorem shows that all this structure is encapsulated by a complex bi-Lagrangian structure.

Theorem 3.8. *Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold. Let M be a complexification of N. Then M has a canonical complex bi-Lagrangian structure in a neighborhood of N. More precisely, in a sufficiently small connected open neighborhood of N in M, there exists a unique complex bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ such that :*

(i) The complex symplectic structure ω is the holomorphic extension of ω_0:

$$\omega = \omega_0^c \ .$$

(ii) The transverse complex Lagrangian foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are the canonical foliations of M defined in Definition 3.6.

(iii) The complex bi-Lagrangian metric g (see § 2.7) is $-i$ times the holomorphic extension of g_0:

$$g = -i\, g_0^c \ .$$

Proof. It is clear that we can take a neighborhood U of N in M small enough so that g_0^c, I_0^c, and ω_0^c, as well as $\mathcal{F}_1$ and $\mathcal{F}_2$, are all well-defined (and uniquely defined) in U. In order to show that $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is a complex bi-Lagrangian structure, all that is left to show is that the canonical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are isotropic for $\omega = \omega_0^c$. As we are about to see, this is very straightforward in local holomorphic coordinates. Let $(z^1, \ldots, z^n)$ thus be a system of local holomorphic coordinates on N. Since the Kähler form ω_0 is of type $(1,1)$, it can be written:

$$\omega_0 = -\frac{1}{2i} h_{j\bar{k}} \, \mathrm{d}z^j \wedge \overline{\mathrm{d}z^k}$$

where $h_{j\bar{k}}$ are real-analytic complex-valued functions on N which satisfy $h_{k\bar{j}} = \overline{h_{j\bar{k}}}$. Each of these functions admits a unique holomorphic extension $h^c_{j\bar{k}}$ (provided U is small enough), obtained by holomorphically extending both the real and imaginary parts of $h_{j\bar{k}}$. Denote by $(z^1, \ldots, z^n, w^1, \ldots, w^n)$ the local holomorphic coordinates in M as in § 3.2. Then the holomorphic extension ω_0^c of ω_0 is simply given by

$$\omega_0^c = -\frac{1}{2i} h^c_{j\bar{k}} \, \mathrm{d}z^j \wedge \mathrm{d}w^k =: \omega \, .$$

Indeed, by uniqueness of the holomorphic extension, it is enough to check that:

(a) ω is a complex symplectic structure. This is immediate, because $(z^1, \ldots, z^n, w^1, \ldots, w^n)$ are holomorphic coordinates and $h^c_{j\bar{k}}$ are holomorphic functions.

(b) ω restricts to ω_0 in $N \subset M$. This is also immediate, because $w^i = \overline{z^i}$ and $h^c_{j\bar{k}} = h_{j\bar{k}}$ at points of $N \subset M$.

Now recall that the canonical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of M are defined by $\{z = \mathit{constant}\}$ and $\{w = \mathit{constant}\}$ respectively. With the expression of ω_0^c above, it is immediate that both these foliations are isotropic for ω. This concludes the proof of (i) and (ii). We delay proving (iii) until the proof of Theorem 3.13. □

Observe in particular that the canonical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are not only isotropic for the holomorphic metric, but also totally geodesic and flat (see § 2.3). Let us give the following corollary as a simple example of consequence.

Corollary 3.9. *Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold. Assume that the holomorphic extension $\omega = \omega_0^c$ exists everywhere in the canonical complexification $N^c = N \times \overline{N}$. Then N admits a natural family of complex affine structures, parametrized by the points of N.*

This is just a consequence of the fact that N can be identified to any horizontal leaf in the product $N \times \overline{N}$, and the leaf space is parametrized by N. Note that these affine structures have no reason to be complete in general.

Remark 3.10. The existence of a flat connection in a compact complex manifold implies that all Chern classes of the tangent bundle vanish, in particular the Euler characteristic vanishes as it identifies with the top Chern class. The next corollary follows.

Corollary 3.11. *Let N be a compact complex manifold which admits a Kähler form whose holomorphic extension exists everywhere in the canonical complexification $N^c = N \times \overline{N}$. Then N has zero Euler characteristic.*

Note that recently, paracomplex geometry has been applied to prove a famous conjecture of Chern on the vanishing Euler characteristic of closed affine manifolds with parallel volume [Kli17].

3.4 The bicomplex Kähler structure

In this subsection we show how the bicomplex and bicomplex Kähler structures introduced in § 2.6 provide clarity on the complexification of complex and Kähler manifolds.

3.4.1 Bicomplex structure in the complexification of a complex manifold

Let (N, I_0) be a complex manifold and (M, J) a complexification of N. Denote $I := I_0^c$ the holomorphic extension of I_0 (cf Proposition 3.4).

Theorem 3.12. *The triple $(I, J, F := IJ)$ is a well-defined holomorphic bicomplex structure in a neighborhood of N in M. Moreover:*

(i) F is the integrable para-complex complex structure associated to the canonical pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ of Definition 3.6.

(ii) This holomorphic bicomplex structure is defined everywhere in the canonical complexification $M = N^c = N \times \overline{N}$.

Proof. The fact that I and J commute and that the ± 1-eigendistributions of $F = IJ$ are the vertical and horizontal foliations of M is a straightforward calculation in the local coordinates (z^i, w^i) introduced in § 3.2, we leave this calculation to the reader. It follows that (I, J, F) is a bicomplex structure, moreover it is holomorphic because I is holomorphic on (M, J) (alternatively: the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are holomorphic). Since J as well as the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ exist everywhere in $M = N^c$, so do F and $I = -JF$. □

3.4.2 Bicomplex Kähler structure in the complexification of a Kähler manifold

Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold and (M, J) a complexification of N. Denote $I := I_0^c$, $ig := g_0^c$, and $\omega := \omega_0^c$ the holomorphic extensions of I_0, g_0, and ω_0 respectively.

Theorem 3.13. *The triple $(g, F := IJ, \omega)$ defines a holomorphic bicomplex Kähler structure in (a neighborhood of N in) M. Moreover, the associated complex bi-Lagrangian structure (cf Proposition 2.15) is the canonical bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ (cf Theorem 3.8).*

Proof. We already know that (I, J, F) is a holomorphic bicomplex structure in M whose associated pair of foliations is $(\mathcal{F}_1, \mathcal{F}_2)$ by Theorem 3.12, and that $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is a complex bi-Lagrangian structure by Theorem 3.8. The only thing left to show is that g is the complex bi-Lagrangian metric, in other words that the identity $\omega(u, v) = g(Fu, v)$ holds. This is a quick and satisfying computation:

$$\begin{aligned}
g(Fu, v) &= -i g_0^c(Fu, v) && \text{(by definition of } g) \\
&= -i g_0^c(-JIu, v) && \text{(since } F = -IJ = -JI) \\
&= g_0^c(Iu, v) && \text{(since } g_0^c \text{ is } J\text{-complex linear)} \\
&= \omega_0^c(u, v) && \text{(see argument below)}
\end{aligned}$$

The last line is justified by analytic continuation: since $g_0(I\cdot, \cdot) = \omega_0(\cdot, \cdot)$ holds in $N \hookrightarrow M$, the identity $g_0^c(I\cdot, \cdot) = \omega_0^c(\cdot, \cdot)$ must hold everywhere defined in M. □

We refer to Figure 1 in the Introduction for a partial overview the interrelationships between all the structures discussed so far.

3.5 Example: $\mathbb{C}P^1$

Let $N = \mathbb{C}P^1$ be the complex projective line. Let $N^c = \mathbb{C}P^1 \times \overline{\mathbb{C}P^1}$ denote the canonical complexification of N. Let z denote the usual complex coordinate in the affine patch $U := \mathbb{C}P^1 \setminus \{[1:0]\}$, and let $w = \overline{z}$ in a copy of U. Then (z, w) are holomorphic coordinates in $U \times \overline{U} \subset N^c$. The manifold $N = \mathbb{C}P^1$ sits inside N^c as the totally real locus $w = \overline{z}$. The canonical foliations in $U \times \overline{U}$ are the vertical and horizontal foliations of $U \times \overline{U} \approx \mathbb{C} \times \overline{\mathbb{C}}$, given by $\{z = \mathit{constant}\}$ and $\{w = \mathit{constant}\}$ respectively.

3.5.1 Complexification of the Fubini-Study Kähler structure

The complex projective line $\mathbb{C}P^1$ has a natural *Fubini-Study* Kähler structure inherited from the flat Kähler structure of $\mathbb{C}^2$. Indeed, $\mathbb{C}P^1$ can be described as the quotient of $S^3 \subset \mathbb{C}^2$ by the isometric action of U(1) by multiplication (this yields the Hopf fibration $S^1 \to S^3 \to S^2$). In the affine coordinate z, the Fubini-Study Kähler structure (g_0, I_0, ω_0) is given by

$$\begin{aligned} g_0 &= \frac{\mathrm{d}z\ \mathrm{d}\bar{z}}{\left(1+|z|^2\right)^2} \\ I_0 &= i\ \mathrm{d}z \otimes \frac{\partial}{\partial z} - i\ \mathrm{d}\bar{z} \otimes \frac{\partial}{\partial \bar{z}} \\ \omega_0 &= \frac{i\ \mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\left(1+|z|^2\right)^2}\ . \end{aligned}$$

The holomorphic extensions of these tensor fields in N^c are the holomorphic metric g_0^c, the complex linear endomorphism I_0^c and the complex symplectic form ω_0^c given by:

$$\begin{aligned} g_0^c &= \frac{\mathrm{d}z\ \mathrm{d}w}{(1+zw)^2} \\ I_0^c &= i\ \mathrm{d}z \otimes \frac{\partial}{\partial z} - i\ \mathrm{d}w \otimes \frac{\partial}{\partial w} \\ \omega_0^c &= \frac{i\ \mathrm{d}z \wedge \mathrm{d}w}{2\,(1+zw)^2} \end{aligned}$$

Observe that g_0^c and ω_0^c are only defined in a neighborhood of $N = \{\bar{z} = w\}$ in N^c: they are singular at points where $1 + zw = 0$.

3.5.2 Bi-Lagrangian structure and connection

The canonical foliations of N^c are the vertical and horizontal foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathbb{C}P^1 \times \overline{\mathbb{C}P^1}$. Let us call $\left(\mathcal{F}_1^{z_0}\right)$ (for $z_0 \in \overline{\mathbb{C}P^1}$) and $\left(\mathcal{F}_2^{w_0}\right)$ (for $w_0 \in \mathbb{C}P^1$) the leaves of the foliations, respectively given by $\mathcal{F}_1^{z_0} = \{(z,w) \in N^c : z = z_0\}$ and $\mathcal{F}_2^{w_0} = \{(z,w) \in N^c : w = w_0\}$. It is clear that these leaves are complex Lagrangian for ω_0^c.

It is easy to check in this scenario that the complex bi-Lagrangian metric g is equal to $-ig_0^c$ as predicted by Theorem 3.8 (iii), indeed: $g = \omega_0^c(F\cdot,\cdot)$ where $F = \mathrm{pr}_1 - \mathrm{pr}_2$, that is:

$$g = \frac{-i\ \mathrm{d}z\ \mathrm{d}w}{(1+zw)^2} = -i\, g_0^c\ .$$

We now turn to the complex bi-Lagrangian connection ∇, which we compute as the Levi-Civita connection of g using *Cartan's structural equations*. We choose a g-orthonormal frame (E_1, E_2) in the holomorphic tangent bundle of N^c and compute its dual coframe (χ^1, χ^2):

$$\begin{aligned} E_1 &= \alpha\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right) \qquad & \chi^1 &= \frac{\mathrm{d}z + \mathrm{d}w}{2\alpha} \\ E_2 &= i\alpha\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial w}\right) \qquad & \chi^2 &= \frac{\mathrm{d}z - \mathrm{d}w}{2i\alpha} \end{aligned}$$

where

$$\alpha = \frac{1+i}{\sqrt{2}}(1+zw)\ .$$

The connection 1-forms ω_2^1 and $\omega_1^2 = -\omega_2^1$ must satisfy $\mathrm{d}\chi^1 = \chi^2 \wedge \omega_2^1$ and $\mathrm{d}\chi^2 = \chi^1 \wedge \omega_1^2$ according to Cartan's first structural equations, which yields

$$\omega_2^1 = \frac{-2i}{1+zw}(w\,\mathrm{d}z - z\,\mathrm{d}w)\,.$$

Note that $d\omega_2^1 = 4\omega_0^c$ as expected from Cartan's second structural equation, knowing that the Fubini-Study metric has constant sectional curvature 4. The connection ∇ is then determined by $\nabla E_1 = \omega_1^2 \otimes E_2$ and $\nabla E_2 = \omega_2^1 \otimes E_1$. In the (z, w) coordinates, ∇ is thus given by:

$$\begin{aligned} \nabla\frac{\partial}{\partial z} &= \frac{-2w}{1+zw}\,\mathrm{d}z \otimes \frac{\partial}{\partial z} & \qquad \nabla\frac{\partial}{\partial \overline{z}} &= \overline{\nabla\frac{\partial}{\partial z}} = \frac{-2\overline{w}}{1+\overline{zw}}\,\mathrm{d}\overline{z} \otimes \frac{\partial}{\partial \overline{z}} \\ \nabla\frac{\partial}{\partial w} &= \frac{-2z}{1+zw}\,\mathrm{d}w \otimes \frac{\partial}{\partial w} & \qquad \nabla\frac{\partial}{\partial \overline{w}} &= \overline{\nabla\frac{\partial}{\partial w}} = \frac{-2\overline{z}}{1+\overline{zw}}\,\mathrm{d}\overline{w} \otimes \frac{\partial}{\partial \overline{w}}\,. \end{aligned} \tag{4}$$

3.5.3 Geodesics and affine structure in the vertical leaves

Let $L = \mathcal{F}_1^{z_0}$ be a vertical leaf. Let us compute the geodesics in L for the Bott connection. These are also geodesics for the bi-Lagrangian connection ∇, since ∇ restricts to the Bott connection in L (in other words, the leaves are totally geodesic for the bi-Lagrangian connection). Let $\gamma(t) = (z_0, w(t))$ be a path in L. It is a geodesic if and only if $\nabla_t \gamma(t) = 0$. Using the expression of ∇ found above (equation (4)), this reduces to the equation $w''(t) - \dfrac{2z_0 w'(t)^2}{1+z_0 w(t)} = 0$. This ODE is quickly solved noting that it is rewritten $f''(t) = 0$, where $f(t) = \dfrac{1}{1+z_0 w(t)}$. The following proposition follows.

Proposition 3.14. *The geodesic $\gamma(t)$ in L (or in N^c) with initial value $\gamma(0) = (z_0, w_0)$ and initial tangent vector $\gamma'(0) = a\dfrac{\partial}{\partial w} + \overline{a}\dfrac{\partial}{\partial \overline{w}}$ is given by $\gamma(t) = (z_0, w(t))$ with $w(t) = \dfrac{at + w_0(1+z_0 w_0)}{-z_0 a t + 1 + z_0 w_0}$.*

We can then proceed to describe the complex affine structure in the leaf. The exponential map at a point (z_0, w_0) for the Bott connection in L identifies (an open set of) the tangent space $\mathrm{T}_{(z_0,w_0)}L$ with (an open set of) L as affine spaces. We choose $w_0 = 0$ and identify $\mathrm{T}_{(z_0,0)}L \approx \mathbb{C}$ via $a\dfrac{\partial}{\partial w} + \overline{a}\dfrac{\partial}{\partial \overline{w}} \mapsto a$. The affine structure in L (thought of as a $\mathbb{C}$-action on L) is then given by, for $(z_0, w) \in L$ and $a \in \mathbb{C}$:

$$(z_0, w) * a = (z_0, f_a(w))$$

where f_a is the projective linear transformation (homography) associated to the matrix

$$M_a = \begin{pmatrix} 1 + a z_0 & a \\ -a {z_0}^2 & 1 - a z_0 \end{pmatrix}.$$

One can check that $M_a = P_{z_0} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} P_{z_0}^{-1}$ where $P_{z_0} = \begin{pmatrix} z_0 & 1 \\ -{z_0}^2 & 0 \end{pmatrix}$. The following proposition ensues.

Proposition 3.15. *The affine structure in the leaf $L = \mathcal{F}_1^{z_0} \approx \mathbb{C}P^1$, in the coordinate w, is the affine structure identifying $\mathbb{C}P^1$ (minus a point) with the affine linear space $\mathbb{C}$ via the map*

$$\begin{aligned} \mathbb{C}P^1 &\to \mathbb{C} \\ w &\mapsto \frac{-1}{z_0(1+z_0 w)}\,. \end{aligned}$$

Note that the affine structure is singular at the locus $1 + z_0 w = 0$, which is not surprising as we saw that the bi-Lagrangian structure is singular there. In particular, the bi-Lagrangian connection is incomplete.

4 Background on Teichmüller theory

In this section we provide a brief introduction to Teichmüller theory and the theory of deformation spaces associated to surfaces. In particular, we will review:

- The Teichmüller space $\mathcal{T}(S)$ of a surface S, parametrizing isotopy classes of complex structures on S,
- The Fricke-Klein deformation space $\mathcal{F}(S)$ of isotopy classes of hyperbolic structures on S,
- The deformation space $\mathcal{CP}(S)$ of isotopy classes of complex projective structures on S,
- Fuchsian space $\mathcal{F}(S)$ and quasi-Fuchsian space $\mathcal{QF}(S)$, parametrizing conjugacy classes of Fuchsian groups isomorphic to $\pi_1(S)$ and their quasiconformal deformations,
- The character variety $\mathcal{X}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))$, parametrizing conjugacy classes of group homomorphisms $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$.

In the next section, we will see how to study these deformation spaces using symplectic geometry and in particular bi-Lagrangian structures.

The primary purpose of this section is to establish the context and notations that we will rely on in the next section. However, the reader who is familiar with Teichmüller theory and the aforementioned deformation spaces may safely skip this section and return to it whenever necessary.

Throughout this section, S is a connected, oriented, smooth, closed surface of negative Euler characteristic. We let g denote its genus ($g \geqslant 2$) and π its fundamental group with respect to some fixed basepoint.

4.1 Geometric structures and character varieties

We start by introducing the general notion of locally homogeneous geometric structures on manifolds, their attendant deformation spaces, and the corresponding holonomy maps to character varieties. We refer to [Gol10] for a nice survey of geometric structures (also see [Thu97], [Gol88] and [Bau14]).

4.1.1 Geometric structures and their deformation space

Let $X = G/H$ be a homogeneous space for the action of a Lie group G. More precisely, assume that X is a real-analytic manifold on which a real Lie group G acts real-analytically and transitively. In the language of geometric structures, the pair (X, G) is called a *(Klein) geometry*. An (X, G)-*structure* on a smooth manifold M is given by an (equivalence class of / maximal) atlas of charts mapping open sets of M into X such that the transition functions coincide with the action of elements of G on X. Such atlases are required to be compatible with the smooth structure on M and they must also respect the orientation of M if M is oriented.

The group $\mathrm{Diff}(M)$ of (orientation-preserving if M is oriented) diffeomorphisms of M has a right action on the set of all (X, G)-structures on M by pulling back atlases. Let $\mathrm{Diff}_0(M) < \mathrm{Diff}(M)$ denote its identity component, consisting of all diffeomorphisms of M isotopic to the identity map. The *deformation space of (marked)* (X, G)-*structures* on M is defined as the quotient of the space of all (X, G)-structures by the action of $\mathrm{Diff}_0(M)$:

$$\mathcal{D}_{(X,G)}(M) = \{(X, G)\text{-structures on } M\} \Big/ \mathrm{Diff}_0(M) \ .$$

This deformation space comes with a natural topology (see [Gol88] for details). In general it may be quite pathological (for instance, the deformation space of affine structures on a 2-torus is not Hausdorff, see [BG05]), but in this paper we will study geometries for which the deformation space is a real-analytic manifold.

Remark 4.1. The *mapping class group* $\pi_0(\mathrm{Diff}(M)) = \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ (also called the *diffeotopy group* of M) naturally acts on the deformation space, and the quotient space

$$\mathcal{M}_{(X,G)}(M) = \mathcal{D}_{(X,G)}(M) \Big/ \pi_0(\mathrm{Diff}(M)) = \{(X,G)\text{-structures on } M\} \Big/ \mathrm{Diff}(M)$$

may be called the *moduli space of* (X,G)*-structures* on M.

Remark 4.2. Some authors prefer to use the group $\mathrm{Diff}_1(M)$ of *homotopically trivial* diffeomorphisms instead of $\mathrm{Diff}_0(M)$ in the definition of the deformation space, see [Bau14, Remark 3.19]. When $M = S$ is a surface, $\mathrm{Diff}_0(S)$ and $\mathrm{Diff}_1(S)$ coincide, so the distinction does not matter. In general $\mathrm{Diff}_0(M)$ and $\mathrm{Diff}_1(M)$ may be different but the former is always a normal subgroup of the latter with discrete quotient (see [Hat16] for more details).

Holonomy

A useful way to construct (X,G)-structures is to find discrete groups $\Gamma < G$ acting freely and properly on open sets $\Omega \subseteq X$. Indeed, in that scenario the quotient $M := \Omega/\Gamma$ inherits a unique (X,G)-structure such that the covering map $\Omega \to M$ is a morphism of (X,G)-manifolds. Such (X,G)-structures are called *embedded*. Now assume M is a fixed smooth manifold and denote by π its fundamental group with respect to some fixed basepoint. Any embedded (X,G)-structure on M can be described by a group homomorphism $\rho : \pi \to G$ with discrete image Γ and a covering map $f : \tilde{M} \to \Omega \subseteq X$ such that f is ρ*-equivariant*, *i.e.* equivariant for the action of π on the universal cover $\tilde{M}$ by deck transformations and on X via $\rho : \pi \to G$. It follows from the equivariance of f that there is a quotient map $\overline{f} : M \to \Omega/\Gamma$ which identifies M to Ω/Γ as (X,G)-manifolds.

More generally, any (X,G)-structure on M can be described by a *developing map* $f : \tilde{M} \to X$ and a *holonomy representation* $\rho\colon \pi \to G$, where f is a ρ-equivariant local diffeomorphism for some group homomorphism ρ. The developing map can be thought of as a global chart on $\tilde{M}$, conversely a developing map for a given (X,G)-structure on M can be reconstructed by analytic continuation starting with any local chart (refer to e.g. [Thu97] for a more detailed description). The development pair (f,ρ) for a given (X,G)-structure is unique up to the action of G on such pairs by post-composition on f and conjugation on ρ. Note that embedded (X,G)-structures are exactly those whose developing map is a covering onto its image. Embedded (X,G)-structures with a developing map which is a covering onto X are called *complete*.

The diffeomorphism group $\mathrm{Diff}(M)$ has a natural right action on the set $\mathrm{Dev}_{(X,G)}(M)$ of all development pairs (f,ρ) by precomposition on f. This action commutes with the left action of G on such pairs (by post-composition on f and conjugation on ρ). Consequently, assigning to a marked (X,G)-structure the conjugacy class of its holonomy representation is a well-defined operation that induces a so-called *holonomy map*

$$hol\colon \mathcal{D}_{(X,G)}(M) \to \mathrm{Hom}(\pi,G)/G\ . \tag{5}$$

In general, this map can be somewhat corrupt: one would like to say that it is a local homeomorphism but that is not always true, see Remark 4.3 below. However, we shall see that it is well-behaved for the geometries that we consider in this paper.

Remark 4.3. The best positive result about the holonomy map in general is that its G-equivariant lift

$$\widehat{hol}\colon \mathrm{Dev}_{(X,G)}(M) \Big/ \mathrm{Diff}_0(M) \to \mathrm{Hom}(\pi,G)$$

is a local homeomorphism (*Deformation theorem*, see [Gol10]). The quotient map *hol* is still open, but local injectivity may fail in general (see [Kap90], [Bau14]). However, in restriction to the *stable* locus corresponding to (X, G)-structures with irreducible holonomy (see § 4.1.2 below), the action of G is proper, so that *hol* descends to a local homeomorphism at least away from the fixed points of the action (which correspond to the orbifold points of the stable locus in the character variety).

After defining character varieties in § 4.1.2 below, we will preferably call holonomy map the map

$$hol\colon \mathcal{D}_{(X,G)}(M) \to \mathcal{X}(\pi, G) \tag{6}$$

obtained by composition of the holonomy map (5) with the natural surjection $\mathrm{Hom}(\pi, G)/G \to \mathcal{X}(\pi, G)$.

4.1.2 Character varieties

The quotient $\mathrm{Hom}(\pi, G)/G$ is usually quite pathological: $\mathrm{Hom}(\pi, G)$ can already be singular, but worse is the fact that the action of G on it by conjugation is generally neither proper nor free. For that reason, it is often preferable to work with a Hausdorff quotient called the *character variety* and denoted $\mathcal{X}(\pi, G) = \mathrm{Hom}(\pi, G)/\!/G$, which moreover has an algebraic structure when G is algebraic. Let us discuss this a bit more precisely in what follows and refer to [Sik12] for more details.

Let π be a finitely generated group and let G be a reductive complex algebraic group[5], for instance $G = \mathrm{PSL}_n(\mathbb{C})$. Let $\mathrm{Hom}(\pi, G)$ denote the space of group homomorphisms (also abusively called "representations") from π to G. It is an affine algebraic set on which G acts by conjugation, and the character variety $\mathcal{X}(\pi, G)$ is defined as the algebraic quotient (or *GIT quotient*)[6]:

$$\mathcal{X}(\pi, G) = \mathrm{Hom}(\pi, G)/\!/_{G}\,.$$

The character variety $\mathcal{X}(\pi, G)$ is an affine algebraic set (but it is not necessarily irreducible when $\mathrm{Hom}(\pi, G)$ itself is not). As a topological space, it is the largest Hausdorff quotient of $\mathrm{Hom}(\pi, G)/G$, in other words

$$\mathcal{X}(\pi, G) = \mathrm{Hom}(\pi, G)\,/_{\sim} \tag{7}$$

where the equivalence relation $\sim$ on $\mathrm{Hom}(\pi, G)$ identifies homomorphisms whose orbit closures intersect. Note that (7) may be taken as a definition of the character variety when G is merely a real Lie group (or in fact a Hausdorff topological group), for instance $G = \mathrm{PSL}_n(\mathbb{R})$. When G is a reductive complex algebraic group, any equivalence class contains a unique closed orbit, and representations $\rho : \pi \to G$ with closed orbits are precisely the completely reducible representations[7]. Such representations are the *polystable* locus $\mathrm{Hom}^{\mathrm{ps}}(\pi, G) \subset \mathrm{Hom}(\pi, G)$ in the language of Geometric Invariant Theory. Therefore, the character variety (for G reductive algebraic) may be defined as a topological space as

$$\mathcal{X}(\pi, G) = \mathrm{Hom}(\pi, G)\,/_{\sim} = \mathrm{Hom}^{\mathrm{ps}}(\pi, G)\,/_{G}\,.$$

[5]A reductive algebraic group is assumed affine by definition (equivalently it is a linear group, according to a well-known theorem). A connected affine algebraic group over the complex numbers is reductive if and only if it has a reductive Lie algebra and its center is of multiplicative type. It is also equivalent to the group being a complexification of a compact real algebraic group.

[6]Let us quickly recall how this is defined. Let M be an affine algebraic set and G a reductive complex algebraic group acting on M. Let $\mathbb{C}[M]$ denote the coordinate ring of regular functions on M and $\mathbb{C}[M]^G \subset \mathbb{C}[M]$ the subalgebra of G-invariant functions. $\mathbb{C}[M]^G$ is finitely generated provided G is reductive by Nagata's theorem solving Hilbert's 14$^{\text{th}}$ problem. Therefore $\mathbb{C}[M]^G$ is the coordinate ring of an affine algebraic set (namely $\operatorname{Spec} \mathbb{C}[M]^G$). This affine set is called the *(affine) GIT quotient* of M and denoted $M/\!/G$.

[7]By definition, a *completely reducible* representation $\rho : \pi \to G$ is one for which $H := \rho(\pi)$ is a completely reducible subgroup of G, meaning that its Zariski closure is a reductive subgroup. Equivalently, for every parabolic subgroup P containing H, there is a Levi factor of P containing H.

In particular, there is a natural surjective map $\mathrm{Hom}(\pi, G)/G \to \mathcal{X}(\pi, G)$ which restricts to a bijection (in fact a homeomorphism) on $\mathrm{Hom}^{\mathrm{ps}}(\pi, G)/G$.

The *stable* locus $\mathrm{Hom}^{\mathrm{s}}(\pi, G)$ consists of the irreducible representations $\rho : \pi \to G$[8]. When $\pi = \pi_1(S, \cdot)$ is the fundamental group of a closed surface of negative Euler characteristic, all stable points ρ are smooth points of $\mathrm{Hom}(\pi, G)$, so that they project to either smooth points or orbifold points in the character variety $\mathcal{X}(\pi, G)$, depending on whether their centralizer in G is equal to the center $Z(G) < G$ or a finite extension of it. It is additionally known that there are no orbifold points for $G = \mathrm{GL}_n(\mathbb{C})$ or $G = \mathrm{SL}_n(\mathbb{C})$. We refer to [Sik12] for these results and more details.

4.2 Fricke-Klein space $\mathcal{F}(S)$ and the character variety $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{R}))$

We now specialize to 2-dimensional hyperbolic geometry, *i.e.* the Klein geometry (X, G) where $X = \mathbb{H}^2$ is the hyperbolic plane and $G = \mathrm{Isom}^+(\mathbb{H}^2) \approx \mathrm{PSL}_2(\mathbb{R})$[9] is the group of orientation-preserving isometries of the hyperbolic plane. Recall that S is a smooth, closed, oriented, connected surface of negative Euler characteristic.

An (X, G)-structure on S is called a *hyperbolic structure*, and the deformation space

$$\mathcal{F}(S) = \mathcal{D}_{(X,G)}(M) = \{\text{hyperbolic structures on } M\} \Big/ \mathrm{Diff}_0(M)$$

is called the *Fricke-Klein space* of S.

One can show that $\mathcal{F}(S)$ is topologically a cell of dimension $6g - 6$ where g is the genus of S, and that it has a natural real-analytic structure (see [Pap07] for several different proofs).

In this setting, all (X, G)-structures are automatically complete and the holonomy map (6) is an embedding onto a connected component of the character variety:

Theorem 4.4 (Goldman [Gol80])**.** *The holonomy map hol* : $\mathcal{F}(S) \to \mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{R}))$ *is an embedding. Its image in the character variety is a connected component, consisting precisely of the equivalence classes of all discrete and faithful group homomorphisms* $\rho \colon \pi \to G$[10].

It is a general fact that if G preserves a Riemannian metric on X (equivalently G acts on X with compact stabilizers) and M is closed, then all (X, G)-structures are complete. Furthermore, under some topological restrictions on M (cf. see Remark 4.2), the holonomy map $hol \colon \mathcal{D}_{(X,G)}(M) \to \mathcal{X}(\pi, G)$ is an embedding. Goldman showed that in the specialization being discussed here, the image of $\mathcal{F}(S)$ in the character variety is the connected component with maximal Euler class, see [Gol80] for more details. That component of the character variety is often called the "Teichmüller component", but we prefer to call it the Fricke-Klein component.

Remark 4.5. The "character variety" $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{R}))$ does not come with an algebraic structure as described in § 4.1.2 because $\mathrm{PSL}_2(\mathbb{R})$ is not a complex reductive algebraic group, but $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{R}))$ is still well-defined at least as a Hausdorff quotient. That being said, there is of course a natural "inclusion"[11] map $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{R})) \to \mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$, and the complex character variety $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ does come with an algebraic structure, but the image of $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{R}))$ in $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ is only a real *semi*-algebraic subset in general. Daniele Alessandrini pointed out to us that it is also a real-analytic subvariety in the case where $\pi = \pi_1(S, \cdot)$ is the fundamental group of a closed surface by

[8] A representation $\rho : \pi \to G$ is called *irreducible* when $\rho(\pi)$ is not contained in any parabolic subgroup of G. Technically, irreducible representations are the stable points of $\mathrm{Hom}(\pi, G)$ for the action of $G/Z(G)$, they are only ("properly") stable for the action of G when G has finite center.

[9] Some readers might prefer to think of it as $G = \mathrm{PSU}(1, 1)$, $G = \mathrm{SO}_0(1, 2)$, $G = \mathrm{PSp}(2, \mathbb{R})$ or $G = \mathrm{PGL}^+(2, \mathbb{R})$.

[10] To be accurate, the discrete and faithful representations form *two* connected components of the real character variety. They correspond to holonomies of hyperbolic structures on S equipped with either of its two possible orientations.

[11] This "inclusion" is actually a $2:1$ map, essentially because $\mathrm{PSL}_2(\mathbb{R}) \neq \mathrm{PGL}_2(\mathbb{R})$.

virtue of the *nonabelian Hodge correspondence*[12] and Hitchin's parametrization of $\mathrm{SL}_2(\mathbb{R})$-Higgs bundles (see [Hit87, section 7]).

4.3 Teichmüller space $\mathcal{T}(S)$ and the Weil-Petersson metric

The *Teichmüller space* of S is the deformation space of complex structures on S (compatible with its orientation). It is defined the same way as a deformation space of (X, G)-structures (see § 4.1.1), with the only difference that the pseudogroup of invertible holomorphic functions between open sets of $\mathbb{C}$ plays the role of the Lie group G acting on $\mathbb{C}$. Let us make this more precise in what follows.

A *complex structure* on S is given by an (equivalence class of / maximal) atlas of charts mapping open sets of S into the complex line $\mathbb{C}$ such that the transition functions are holomorphic. We typically denote by X the surface S equipped with a complex structure and X is called a *Riemann surface*. In what follows, we always assume that complex structures on S are compatible with its given smooth structure and orientation. The group $\mathrm{Diff}(S)$ of orientation-preserving diffeomorphisms of S naturally acts on complex structures on S (by pulling back atlases). The *Teichmüller space of S* is the space of all complex structures on S up to the action of the subgroup $\mathrm{Diff}_0(S)$ of isotopically trivial diffeomorphisms:

$$\mathcal{T}(S) = \{\text{complex structures on } S\}/\mathrm{Diff}_0(S)\ .$$

Kodaira-Spencer deformation theory (see [KS58], also [EE69]) shows that $\mathcal{T}(S)$ is a complex manifold with holomorphic tangent space $\mathrm{T}_X\mathcal{T}(S) = \check{H}^1(X, \Theta_X)$, where Θ_X is the sheaf of holomorphic vector fields on X. Via Serre duality, one can identify the holomorphic cotangent space as $\mathrm{T}^*_X\mathcal{T}(S) = H^0(X, K_X^2)$ where K_X is the canonical bundle on X, in other words $\mathrm{T}^*_X\mathcal{T}(S)$ is the space of *holomorphic quadratic differentials* on X (tensors on X of the form $\phi = \varphi(z)dz^2$ with φ holomorphic, in a complex coordinate z). A quick application of the Riemann-Roch theorem shows that $\mathcal{T}(S)$ has complex dimension $3g - 3$, where g is the genus of S.

An immediate consequence of the celebrated Poincaré uniformization theorem (and the fact that any hyperbolic structure on S is complete, see § 4.2) is that there is a 1:1 correspondence between complex structures on S and hyperbolic structures on S. Concretely, one can associate to each complex structure X the hyperbolic structure given by the *Poincaré metric*, *i.e.* the unique conformal metric on X with constant curvature -1. This correspondence passes to the quotient as a real-analytic diffeomorphism:

$$F : \mathcal{T}(S) \to \mathcal{F}(S) \tag{8}$$

where $\mathcal{F}(S)$ is the Fricke-Klein space defined above (see § 4.2). For this reason, $\mathcal{F}(S)$ itself is sometimes referred to as "the Teichmüller space of S", but we will maintain the distinction between $\mathcal{T}(S)$ and $\mathcal{F}(S)$.

The *Weil-Petersson product* of two holomorphic quadratic differentials ϕ and ψ is defined by

$$\langle \phi, \psi \rangle_{\mathrm{WP}} = -\frac{1}{4}\int_X \phi \cdot \sigma^{-1} \cdot \overline{\psi}$$

where σ^{-1} is the dual current of the Poincaré area form σ. This Hermitian inner product on $\mathrm{T}^*_X\mathcal{T}(S) = H^0(X, K_X^2)$ defines by duality a Hermitian metric $\langle \cdot, \cdot \rangle_{\mathrm{WP}}$ on the complex manifold $\mathcal{T}(S)$,

[12] This real-analytic correspondence between the character variety and the moduli space of Higgs bundles is due to Hitchin and Donaldson ([Hit87], [Don87]) and to Corlette and Simpson ([Cor88], [Sim91, Sim92]) in the more general case where π is the fundamental group of a smooth Kähler variety and G is a reductive complex algebraic group.

which turns out to be Kähler, as was first shown by Ahlfors [Ahl61]. It is called the *Weil-Petersson metric* on $\mathcal{T}(S)$ and we shall denote it

$$h_{\mathrm{WP}} = g_{\mathrm{WP}} - i\omega_{\mathrm{WP}}$$

where g_{WP} and ω_{WP} are the Weil-Petersson Riemannian metric and Kähler form respectively.

Good general references for Teichmüller theory and the Weil-Petersson metric include [Hub06], [Pap07], and [Wol10].

4.4 Deformation space of complex projective structures $\mathcal{CP}(S)$

4.4.1 Deformation space and holonomy

Let us now consider "one-dimensional complex projective geometry" (X, G), where $X = \mathbb{C}P^1$ is the complex projective line and $G = \mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$ is the group of automorphisms of $\mathbb{C}P^1$ acting by projective linear transformations. In this situation, an (X, G)-structure on S is called a *complex projective structure*, and the deformation space $\mathcal{CP}(S) := \mathcal{D}_{(X,G)}(S)$ is called the *deformation space of (marked) complex projective structures*. A very good reference for complex projective structures is Dumas' survey [Dum09].

A pleasant feature of the deformation space $\mathcal{CP}(S)$ is that it is a complex manifold. As in the case of Teichmüller space $\mathcal{T}(S)$, one can use deformation theory to express the tangent space of $\mathcal{CP}(S)$ in terms of Čech cohomology, but the description of tangent vectors is not quite as explicit as with $\mathcal{T}(S)$. That being said, $\mathcal{CP}(S)$ has a striking parametrization as a holomorphic affine bundle over $\mathcal{T}(S)$. We briefly describe this so-called *Schwarzian parametrization* in § 4.4.2.

It is not hard to show that the holonomy of a complex projective structure on S is always irreducible, so that the holonomy map (6)

$$\mathrm{hol} : \mathcal{CP}(S) \to \mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$$

takes values in the smooth (stable) locus of the character variety $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$. Moreover, it is a local biholomorphism, as first shown by Hejhal [Hej75], Earle [Ear81] and Hubbard [Hub81]. It is however not a covering map onto its image ([Hej75]). Gallo, Kapovich, and Marden [GKM00] precisely determined the image of the holonomy map: a homomorphism $\rho : \pi \to \mathrm{PSL}_2(\mathbb{C})$ occurs as the holonomy of a complex projective structure if and only if ρ is nonelementary and lifts to $\mathrm{SL}_2(\mathbb{C})$. Here, a representation is called *elementary* if its image is bounded or virtually abelian.

4.4.2 Schwarzian parametrization

First observe that there is a "forgetful projection"

$$p \colon \mathcal{CP}(S) \to \mathcal{T}(S)\,. \tag{9}$$

This is simply because a complex projective atlas is in particular a holomorphic atlas, so any complex projective structure on S defines an underlying complex structure. The map p is onto $\mathcal{T}(S)$, because a right inverse is provided by the Fuchsian section $\sigma_0 : \mathcal{T}(S) \to \mathcal{CP}(S)$, see § 4.5 below. It is also not hard to show that p is holomorphic. It turns out that for any $X \in \mathcal{T}(S)$, the fiber $p^{-1}(X)$ can be equipped with the structure of an affine space modeled on the vector space $H^0(X, K_X^2)$, we explain this in the next paragraph. For now, recall that $H^0(X, K_X^2)$ is the cotangent space $\mathrm{T}_X^*\mathcal{T}(S)$, so that globally $\mathcal{CP}(S)$ is an affine bundle modeled on the vector bundle $\mathrm{T}^*\mathcal{T}(S)$. Since an affine space is not canonically isomorphic to its underlying vector space unless an origin is chosen, $\mathcal{CP}(S)$ is not canonically isomorphic to $\mathrm{T}^*\mathcal{T}(S)$. However, the choice of a "zero section" $\sigma : \mathcal{T}(S) \to \mathcal{CP}(S)$ does define an isomorphism $\tau_\sigma : \mathcal{CP}(S) \to \mathrm{T}^*\mathcal{T}(S)$, characterized as the unique isomorphism of affine bundles such that $\tau_\sigma \circ \sigma$ is the zero section of $\mathrm{T}^*\mathcal{T}(S)$.

It remains to explain why, given a point $X \in \mathcal{T}(S)$, the fiber $p^{-1}(X) \subset \mathcal{CP}(S)$ enjoys the structure of an affine space modeled on the vector space $H^0(X, K_X^2)$: how can one associate to two complex projective structures $Z_1, Z_2 \in p^{-1}(X)$ a holomorphic quadratic differential $\phi \in H^0(X, K_X^2)$ representing the affine subtraction $Z_2 - Z_1$? The *Schwarzian derivative* is a differential operator that holds the answer to that question. Let us say what it is in a few sentences and refer to [Dum09] and [And98] for more details. The Schwarzian derivative is originally defined for a locally injective holomorphic function $f : U \to \mathbb{C}$ (where U is some open subset of $\mathbb{C}$) as

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 .$$

One should think of the Schwarzian derivative of f as the holomorphic quadratic differential $Sf = Sf(z)\,dz^2$ rather than a function, given how it transforms under a projective change of coordinate on $\mathbb{C}P^1$. In a sense that can be made precise, the Schwarzian derivative of f measures how far f is from being a projective linear transformation[13]. Now, given a locally injective holomorphic map $f : Z_1 \to Z_2$ where $Z_1, Z_2 \in P(X)$, one can take the Schwarzian derivative of f in local projective coordinates on Z_1 and Z_2, which yields a holomorphic quadratic differential $\phi \in Q(X)$. This holomorphic quadratic differential, for $f = \mathrm{id}_X$, defines the affine subtraction $Z_2 - Z_1 \in H^0(X, K_X^2)$. The fact that it respects the axioms of affine subtraction (*Weyl's axioms*) is a consequence of the classical properties of the Schwarzian derivative that we will not reiterate here (see [Dum09] for details).

4.5 Fuchsian and quasi-Fuchsian structures

A *Kleinian group* is by definition a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$[14]. The group $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$ can be regarded alternatively as the group of automorphisms of the complex projective line $\mathbb{C}P^1$ acting projectively linearly, or the group of the automorphisms the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \approx S^2$ acting by Möbius transformations, or the group of orientation-preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. These points of view are consistent with each other considering that:

- Stereographic projection identifies the round 2-sphere S^2 with the Riemann sphere $\hat{\mathbb{C}}$, which can also be identified to the complex projective line $\mathbb{C}P^1$ (via the "affine patch" $z \mapsto [z : 1]$).
- S^2 is also the *ideal boundary* of $\mathbb{H}^3$ (in other words its Gromov boundary), and an isometry of $\mathbb{H}^3$ is uniquely determined by its continuous extension on the ideal boundary.

Let Γ be a Kleinian group. A point $x \in \mathbb{C}P^1$ is called a *point of discontinuity* (or a *wandering point*) for Γ if x has a neighborhood $U \subset \mathbb{C}P^1$ such that $gU \cap U = \emptyset$ for all but finitely many $g \in \Gamma$. The *domain of discontinuity* of Γ is the set $\Omega \subset \mathbb{C}P^1$ of all points of discontinuity. One can show that Γ acts properly on Ω, and Ω is the largest open set with that property. The *limit set* of Γ is the set $\Lambda = \mathbb{C}P^1 - \Omega$. One can show that Λ is finite if and only if Γ is elementary (virtually abelian).

[13] One can define the *osculating map to f* as the map $Osc(f) : U \to \mathrm{PSL}_2(\mathbb{C})$, whose value at a point z is the projective linear transformation that best approximates f at z. Then the Darboux derivative of $Osc(f)$ is equal to the Schwarzian derivative of f, suitably interpreted. In particular, it is clear that

$$f \text{ is projective linear } \Leftrightarrow Osc(f) \text{ is constant } \Leftrightarrow Sf = 0 .$$

[14] A Kleinian group is classically required to have a nonempty domain of discontinuity (e.g. for Thurston [Thu80], Maskit [Mas88] or Kapovich [Kap09]), but modern usage tends to allow any discrete group.

When Γ is not elementary, its limit set Λ can be characterized as the smallest Γ-invariant closed subset of $\mathbb{C}P^1$, or the set of accumulation points of the orbit of any point $x \in \mathbb{H}^3 \cup \mathbb{C}P^1$.

A *Fuchsian group* is a Kleinian group conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$. A Fuchsian group can be characterized as a Kleinian group whose limit set Λ is contained in some round circle C, and which preserves both components of $\mathbb{C}P^1 - C$ (which are round disks). More generally, a *quasi-Fuchsian group* is a Kleinian group Γ whose limit set Λ is contained in some topological circle C, and which preserves both components of $\mathbb{C}P^1 - C$ (which are topological disks). A (quasi-)Fuchsian group Γ is called *of the first kind* or *of the second kind* depending on whether $\Lambda = C$ or $\Lambda \subsetneq C$. If Γ is a finitely generated torsion-free Kleinian group, then Γ is quasi-Fuchsian of the first kind if and only if it satisfies the seemingly stronger condition that its action on $\mathbb{C}P^1$ is topologically conjugate to the action of a Fuchsian group of the first kind, *i.e.* there exists a homeomorphism $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $f\Gamma f^{-1}$ is Fuchsian ([Ber70, Theorem 4]). Moreover, f can be taken (in fact is necessarily) quasiconformal in this situation.

Let S be a closed oriented surface and let π denote its fundamental group as in the rest of the section. A representation $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{C})$ is called Fuchsian (resp. quasi-Fuchsian) if ρ is injective and its image $\Gamma = \rho(\pi) \subset \mathrm{PSL}_2(\mathbb{C})$ is a Fuchsian (resp. quasi-Fuchsian) group of the first kind. The *deformation space of Fuchsian* (resp. *quasi-Fuchsian*) structures denoted $\mathcal{F}(S)$ (resp. $\mathcal{QF}(S)$) is the subset of the character variety $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ whose points are the equivalence classes of Fuchsian (resp. quasi-Fuchsian) representations. It is a remarkable fact that $\mathcal{QF}(S)$ is an open subset of the character variety[15], moreover it lies in the smooth locus.

Let ρ be a quasi-Fuchsian representation with image $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$. The domain of discontinuity Ω of Γ is the disjoint union of two invariant topological disks Ω^+ and Ω^- on which Γ acts freely properly, therefore the quotients Ω^+/Γ and Ω^-/Γ are both closed surfaces diffeomorphic to S. For notational comfort let us denote $S^+ = S$ the surface S with its given orientation, and S^- the same surface with the opposite orientation. We can assume that we called $\Omega^\pm$ the component of Ω that has the "same" orientation as $S^\pm$, in the sense that there exists an orientation-preserving homeomorphism $f^\pm\colon S^\pm \to \Omega^\pm/\Gamma$ whose induced map at the level of fundamental groups is precisely ρ. Note that both quotient surfaces $\Omega^\pm/\Gamma$ inherit complex projective structures, being given as a quotient of an open set of $\mathbb{C}P^1$ by a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ acting freely properly. In fact, the lifted map $\tilde{f}^\pm\colon \tilde{S} \to \Omega^\pm$ can be seen as a developing map for a complex projective structure $Z^\pm$ on $S^\pm$ with holonomy ρ: this is a typical example of an embedded geometric structure (see Remark 4.1.1). Such projective structures are called *standard quasi-Fuchsian projective structures*[16]. Thus there is a well-defined map

$$(Z^+, Z^-) : \ \mathcal{QF}(S) \to \mathcal{CP}(S^+) \times \mathcal{CP}(S^-)$$

which associates to the quasi-Fuchsian representation ρ the pair of standard quasi-Fuchsian projective structures $(Z^+(\rho), Z^-(\rho))$. It is not hard to show that this map is holomorphic, in fact note that $Z^\pm$ is a right inverse of the holonomy map $\mathrm{hol}^\pm\colon \mathcal{CP}(S^\pm) \to \mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$. Post-composing the map (Z^+, Z^-) with the "forgetful projection" (9) yields a map $\mathcal{QF}(S) \to \mathcal{T}(S^+) \times \mathcal{T}(S^-)$ which associates to the quasi-Fuchsian representation ρ a pair of Riemann surfaces (X^+, X^-). Bers' *simultaneous*

[15]This fact is essentially due to Bers' simultaneous uniformization theorem discussed in the next paragraph. It is moreover true that $\mathcal{QF}(S)$ is the interior of the subset $\mathcal{AH}(S) \subset \mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ of equivalence classes of discrete and faithful representations. This theorem is due to the work of Ahlfors, Bers, Maskit, Kra, Marden, Thurston and Sullivan [AB60, Ahl64, Ber87, Kra72, Mar74, Sul85]. Let us refer to [CM04, Chapter 7] for an exposition of their theory, which holds more generally for geometrically finite representations. The positive answer to the Bers-Sullivan-Thurston "density conjecture" [NS12, Ohs11] shows that furthermore, $\mathcal{AH}(S)$ is the closure of $\mathcal{QF}(S)$ in the character variety.

[16]There are infinitely many complex projective structures with same quasi-Fuchsian holonomy ρ. The standard one is the only one that is an embedded projective structure (whose developing map is an embedding), all the others have infinite-to-one developing maps. They are called *exotic* quasi-Fuchsian structures and Goldman [Gol87] showed that any one of them can be obtained from the standard one by grafting along an rational lamination. Shinpei Baba gave a generalization of this result for an arbitrary holonomy fiber $\mathrm{hol}^{-1}(\rho) \subset \mathcal{CP}(S)$ [Bab15, Bab13].

uniformization theorem [Ber60a] says that this map is bijective, it is in fact a biholomorphism. Let us denote its inverse by

$$QF(\cdot,\cdot):\ \mathcal{T}(S^+)\times\mathcal{T}(S^-)\to\mathcal{QF}(S)\ .$$

For a fixed $X^-\in\mathcal{T}(S^-)$, the map $QF(\cdot,X^-)\colon\mathcal{T}(S)\to\mathcal{QF}(S)$ is called a *horizontal Bers slice*. Similarly, for $X^+\in\mathcal{T}(S)$, the map $QF(X^+,\cdot)\colon\mathcal{T}(S^-)\to\mathcal{QF}(S)$ is called a *vertical Bers slice*. The quasi-Fuchsian space $\mathcal{QF}(S)$ is thus foliated by horizontal Bers slices and by vertical Bers slices, and these two foliations are transverse. After post-composing with the map $Z^+\colon\mathcal{QF}(S)\to\mathcal{CP}(S)$, a horizontal Bers slice becomes a map $\sigma_{X^-}\colon\mathcal{T}(S)\to\mathcal{CP}(S)$ called a *Bers section*, while a vertical Bers slice becomes a map $B_{X^+}\colon\mathcal{T}(S^-)\to\mathcal{CP}(S)$. By construction, the Bers section σ_{X^-} is a section to the forgetful projection $p\colon\mathcal{CP}(S)\to\mathcal{T}(S)$, while the map B_{X^+} lands in the fiber $p^{-1}(X^+)\subset\mathcal{CP}(S)$. Recall that $p^{-1}(X^+)$ is an affine space modeled on the vector space of holomorphic quadratic differentials $H^0(X,K^2_{X^+})$, as explained in § 4.4.2. One can thus define the *Bers embedding* map b_{X^+} using affine subtraction as follows:

$$\begin{aligned} b_{X^+}\colon\mathcal{T}(S^-)&\to H^0(X^+,K^2_{X^+})\\ X^-&\mapsto B_{X^+}(X^-)-Z_0 \end{aligned}\tag{10}$$

where Z_0 is the standard Fuchsian projective structure $Z_0=\underline{QF(X^+,\overline{X^+})}$. Bers showed that b_{X^+} is a holomorphic embedding of Teichmüller space $\mathcal{T}(S^-)\approx\overline{\mathcal{T}(S)}$ in $H^0(X,K^2_{X^+})\approx\mathbb{C}^{3g-3}$, and moreover has bounded image [Ber60b, Ber61].

A special case of the situation discussed above is when ρ happens to be a Fuchsian representation. In that case, Λ is a round circle and Ω^+ and Ω^- are round disks. In fact, after conjugating ρ, we can assume that $\Gamma\subset\mathrm{PSL}_2(\mathbb{R})$, $\Lambda=\mathbb{R}\cup\{\infty\}$ (extended real line), $\Omega^+=\mathbb{H}$ (upper half-plane)and $\Omega^-=\overline{\mathbb{H}}$ (lower half-plane). Since $\mathbb{H}$ can be identified to the hyperbolic plane (it is Poincaré's half-plane model) and Γ acts freely properly on $\mathbb{H}$ by isometries, the quotient surface $Z^+=\mathbb{H}/\Gamma$ inherits a hyperbolic structure, in addition to a complex projective structure, and ρ is the holonomy of both these geometric structures. We have just described an identification between the Fuchsian deformation space and the Fricke-Klein deformation space of S, which is the reason why we somewhat abusively use the same notation $\mathcal{F}(S)$ for both. Clearly, the Fuchsian space $\mathcal{F}(S)\subset\mathcal{QF}(S)$ arises as the image of the map

$$X\in\mathcal{T}(S)\mapsto QF(X,\overline{X})\in\mathcal{QF}(S)$$

called the *Fuchsian slice*. Note that seen as map from $\mathcal{T}(S)$ to the Fricke-Klein space $\mathcal{F}(S)$, this is nothing else than the uniformization map (8). After composing the Fuchsian slice with the map $Z^+\colon\mathcal{QF}(S)\to\mathcal{CP}(S)$, one obtains a map $\sigma_0\colon\mathcal{T}(S)\to\mathcal{CP}(S)$ called the *Fuchsian section*. By construction, the Fuchsian section is a section to the forgetful projection $p\colon\mathcal{CP}(S)\to\mathcal{T}(S)$.

5 Applications to Teichmüller theory

In this section, we present some applications of the symplectic geometry notions that we developed in sections 1, 2, and 3 to Teichmüller theory, more specifically to the deformation spaces introduced in section 4.

Throughout this section again, S is a connected, oriented, smooth, closed surface of genus $g\geqslant 2$ and fundamental group π.

5.1 Symplectic structure of deformation spaces

Deformation spaces associated to a closed surface tend to have a natural symplectic structure. Goldman established the main reason for that:

Theorem 5.1 (Goldman, [Gol84])**.** *Let G be a complex semisimple algebraic group. The character variety $\mathcal{X}(\pi, G)$ enjoys a natural complex symplectic structure ω_G.*

Goldman shows that the 2-form ω_G is an algebraic tensor on the character variety, defining in particular a complex symplectic structure on the smooth locus. If G is merely a real / complex semisimple Lie group, ω_G is still well-defined as a real / complex symplectic structure on the smooth locus. If G is only assumed reductive, the theorem also holds with the trade-off that the symplectic structure is not univocal, its definition requires a choice. Let us explain in a few words how this symplectic structure is constructed. The Zariski tangent space to the character variety at a point $[\rho]$ is given by

$$\mathrm{T}_{[\rho]}\mathcal{X}(\pi, G) = H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$$

where $\mathfrak{g}$ is the Lie algebra of G. The notation $\mathfrak{g}_{\mathrm{Ad}\circ\rho}$ signifies that $\mathfrak{g}$ is a π-module via $\mathrm{Ad}\circ\rho\colon \pi \to \mathrm{Aut}(\mathfrak{g})$, so it is possible to define the group cohomology of π with coefficients in $\mathfrak{g}$. Let B denote the Killing form of $\mathfrak{g}$, which is nondegenerate because $\mathfrak{g}$ is semisimple. Given two elements α, $\beta \in H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$, one can take their cup product with B as coefficient pairing to obtain an element $\alpha \cup_B \beta \in H^2(\pi, \mathbb{C})$. Since $\pi = \pi_1(S)$ and S is a closed oriented surface, $H^2(\pi, \mathbb{C})$ is isomorphic to the singular cohomology $H^2(S, \mathbb{C})$ which is isomorphic to $\mathbb{C}$ by Poincaré duality. Therefore we have described how to assign a complex number to the pair (α, β). It is clear that this assignment globally defines an algebraic nondegenerate 2-form on the character variety $\mathcal{X}(\pi, G)$. It is not easy to show that ω_G is closed as a 2-form, but it follows from a beautiful argument of symplectic reduction due to Atiyah and Bott [AB83] that Goldman adapted to this setting.

A consequence of Goldman's theorem is that any deformation space of (X, G)-structures on S also enjoys a natural symplectic structure: just pull back the symplectic structure of the character variety by the holonomy map (6). Somewhat abusively, we still denote by ω_G the symplectic structure of the deformation space $\mathcal{D}_{(X,G)}(S)$.

In the same paper [Gol84], Goldman shows that in the case of Fricke-Klein space $\mathcal{F}(S)$, the real symplectic structure ω_G coincides with the Weil-Petersson Kähler form ω_{WP} on Teichmüller space. More precisely, the uniformization map (8) : $(\mathcal{T}(S), \omega_{\mathrm{WP}}) \to (\mathcal{F}(S), \omega_G)$ is a real symplectomorphism.

In the case of complex projective structures, ω_G is a complex symplectic structure in $\mathcal{CP}(S)$. The symplectic geometry of $\mathcal{CP}(S)$ was carefully studied in [Lou15a, Lou15c]. In particular, it is shown in [Lou15a] that the Schwarzian parametrization is a symplectomorphism, improving a result of Kawai [Kaw96]. More precisely:

Theorem 5.2 (Loustau [Lou15a])**.** *Let σ be a section to the projection $p : \mathcal{CP}(S) \to \mathcal{T}(S)$ (9). Let $\tau_\sigma \colon \mathcal{CP}(S) \to \mathrm{T}^*\mathcal{T}(S)$ denote the isomorphism of affine bundles given by the Schwarzian parametrization using σ as the "zero section" (see § 4.4.2). The following are equivalent:*

(i) *σ is Lagrangian: $\sigma^*\omega_G = 0$.*

(ii) *$(\tau_\sigma)^*\omega = -i\,\omega_G$, where ω is the canonical complex symplectic structure on the cotangent bundle $\mathrm{T}^*\mathcal{T}(S)$.*

(iii) *$d(\sigma - \sigma_0) = i\,\omega_{\mathrm{WP}}$, where $\sigma_0 : \mathcal{T}(S) \to \mathcal{CP}(S)$ is the Fuchsian section.*

Moreover, it is known that Bers sections (see § 4.5) and their generalizations including Schottky sections satisfy the conditions of this theorem, it is a consequence of the work of McMullen [McM00], Takhtajan and Teo [TT03] and Loustau [Lou15a]. A consequence of this theorem is that the projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$ is a Lagrangian fibration, a fact that was already known to

Goldman [Gol04]. Steven Kerckhoff also proved part or all of these facts in unpublished work. Let us mention that other recent activity in this subject (proofs and new perspectives on Kawai's theorem) are due to Bertola-Korotkin-Norton [BKN17] and Takhtajan [Tak17].

5.2 Complex bi-Lagrangian geometry of quasi-Fuchsian space

We refer to § 4.5 for the definition of quasi-Fuchsian space $\mathcal{QF}(S)$. As an open subset in the smooth locus of the character variety $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$, quasi-Fuchsian space $\mathcal{QF}(S)$ is a complex manifold of dimension $3g-3$. Moreover $\mathcal{QF}(S)$ comes with a natural complex symplectic structure ω_G, which is just the restriction of Goldman's symplectic structure. Let us mention that it was showed in [Lou15a] that the complex Fenchel-Nielsen coordinates introduced by Kourouniotis [Kou94] and Tan [Tan94], which are global holomorphic coordinates in $\mathcal{QF}(S)$, are Darboux coordinates for the complex symplectic structure ω_G.

It turns out that as a complex symplectic manifold, $(\mathcal{QF}(S), \omega_G)$ is the complexification of the real-analytic symplectic manifold $(\mathcal{T}(S), \omega_{\mathrm{WP}})$:

Proposition 5.3. *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{T}(S) & \xrightarrow{F} & \mathcal{F}(S) \\ \big\downarrow{\scriptstyle\Delta} & & \big\downarrow{\scriptstyle hol} \\ \mathcal{T}(S)\times\overline{\mathcal{T}(S)} & \xrightarrow{QF} & \mathcal{QF}(S). \end{array}$$

where the vertical arrows are complexifications of real-analytic manifolds, and the horizontal arrows are given by (simultaneous) uniformization. Moreover, all maps are (complex) symplectomorphisms.

Remark 5.4. Given the above proposition, we see that *simultaneous uniformization is the complexification of uniformization*. By the uniqueness of complexification, this lemma shows that (independent of simultaneous uniformization) there exists a unique biholomorphism between a neighborhood of the diagonal in $\mathcal{T}(S)\times\overline{\mathcal{T}(S)}$ and a neighborhood of $\mathcal{F}(S)$ in $\mathcal{QF}(S)$.

Proof. The commutativity of the diagram is obvious from the discussions in section 4.5. The diagonal map $\Delta : \mathcal{T}(S) \to \mathcal{T}(S)\times\overline{\mathcal{T}(S)}$ is a complexification since this is the definition of the canonical complexification of the complex manifold $\mathcal{T}(S)$ (cf Proposition 3.5). Moreover, the construction of the complex symplectic structure on $\mathcal{T}(S)\times\overline{\mathcal{T}(S)}$ is via analytic continuation of the Kähler form of $\mathcal{T}(S)$ along the diagonal, therefore Δ is a symplectic embedding by construction.

By the commutativity of the diagram, and the fact that the *Fuchsian slice* $QF\circ\Delta : \mathcal{T}(S) \to \mathcal{QF}(S)$ is the composition of a maximal totally real embedding followed by a biholomorphism, the map $\mathrm{hol} : \mathcal{F}(S) \to \mathcal{QF}(S)$ is a complexification. It was proved by Goldman in [Gol84] (as we mentioned in § 5.1) that hol is a symplectic embedding.

Finally, since $F : \mathcal{T}(S) \to \mathcal{F}(S)$ is a symplectomorphism, the uniqueness of the complex symplectic structure in the complexification implies that QF is a symplectomorphism. This completes the proof.

□

Recall that Teichmüller space $\mathcal{T}(S)$ equipped with the Weil-Petersson metric is a Kähler manifold. We can thus use Theorem 3.8 to show that $\mathcal{QF}(S)$, as the complexification of $\mathcal{T}(S)$, enjoys a natural complex bi-Lagrangian structure. Theorem 3.8 only predicts that this structure exists in some neighborhood of the Fuchsian slice, but we show that it is actually well-defined everywhere in $\mathcal{QF}(S)$. Moreover, it is remarkable that the complex symplectic structure and the two transverse Lagrangian foliations that define the bi-Lagrangian structure are respectively Goldman's symplectic structure and the two foliations of $\mathcal{QF}(S)$ by Bers slices:

Theorem 5.5. *There exists a complex bi-Lagrangian structure* $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ *in quasi-Fuchsian space* $\mathcal{QF}(S)$ *such that:*

(i) The complex symplectic structure ω *is equal to Goldman's symplectic structure* ω_G*.*

(ii) The complex Lagrangian foliations $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are the horizontal and vertical foliations of* $\mathcal{QF}(S)$ *by Bers slices.*

Moreover, this bi-Lagrangian structure coincides with the complex bi-Lagrangian structure predicted by Theorem 3.8, where $\mathcal{QF}(S)$ *is seen as a complexification of Teichmüller space* $\mathcal{T}(S)$ *via the Fuchsian slice* $QF \circ \Delta = \mathrm{hol} \circ F : \mathcal{T}(S) \to \mathcal{QF}(S)$*. (cf Proposition 5.3).*

Proof. We have seen that $\mathcal{QF}(S)$ can be identified holomorphically to $\mathcal{T}(S^+) \times \mathcal{T}(S^-)$ *i.e.* $\mathcal{T}(S) \times \overline{\mathcal{T}(S)}$ via Bers' simultaneous uniformization theorem. Under this identification, the foliations of $\mathcal{QF}(S)$ by horizontal and vertical Bers slices correspond by definition to the horizontal and vertical foliations of the product $\mathcal{T}(S)\times\overline{\mathcal{T}(S)}$. These two foliations are precisely what we called the canonical foliations of the canonical complexification $\mathcal{T}(S) \times \overline{\mathcal{T}(S)}$ (see Definition 3.6), and we showed in Theorem 3.8 that these two foliations are the foliations of the complex bi-Lagrangian structure of the complexification. The fact that the complex symplectic structure coincides with Goldman's symplectic structure was proved in Proposition 5.3 above. The fact that the bi-Lagrangian structure exists everywhere in $\mathcal{QF}(S)$ is an easy consequence of the fact that Goldman's symplectic structure, as well as the two foliations of $\mathcal{QF}(S)$ by Bers slices, exist everywhere in $\mathcal{QF}(S)$. □

An immediate yet interesting consequence of Theorem 5.5 is the existence of a natural holomorphic metric in quasi-Fuchsian space, namely the complex bi-Lagrangian metric. As far as the authors are aware, this holomorphic metric in $\mathcal{QF}(S)$ has not been introduced let alone studied prior to this paper, and suggests that the most natural "metric" structure on $\mathcal{QF}(S)$ is neutral pseudo-Riemannian rather than Riemannian.

Theorem 5.6. *There exists a holomorphic metric* g *defined on quasi-Fuchsian space* $\mathcal{QF}(S)$ *such that:*

(i) g *restricts to* $-ig_{\mathrm{WP}}$ *on the Fuchsian slice* $\mathcal{F}(S)$*, where* g_{WP} *is the Weil-Petersson metric on Teichmüller space* $\mathcal{T}(S)$*, identified to* $\mathcal{F}(S)$ *via uniformization.*

(ii) The two foliations of quasi-Fuchsian space by Bers slices are isotropic for g*, furthermore they are totally geodesic and flat for the real and imaginary parts of* g*.*

(iii) The Levi-Civita connection of g *parallelizes Goldman's complex symplectic structure* ω_G*.*

(iv) Let u*,* v *be tangent vectors at some point of* $\mathcal{QF}(S)$*. Then*

$$g(u, v) = i(\langle q_{u_1}, \mu_{v_2}\rangle + \langle q_{v_1}, \mu_{u_2}\rangle) \tag{11}$$

where $\langle\, , \rangle$ *is the duality pairing between* $\mathrm{T}^*\mathcal{T}(S)$ *and* $\mathrm{T}\mathcal{T}(S)$*, and the notations* q_{u_i}*,* μ_{v_i} *are explained right below this theorem.*

Before writing the proof of Theorem 5.6, let us introduce the notations that we use in (11). First recall that for a tangent vector u to $\mathcal{QF}(S)$, we write $u = u_1 + u_2$, where u_1 (resp. u_2) is tangent to the vertical (resp. horizontal) Bers foliation. Now observe that:

- Since u_2 is tangent to a horizontal Bers slice which is a copy of Teichmüller space, one can identify u_2 to a tangent vector to Teichmüller space. We denote it $\mu_{u_2} \in \mathrm{T}\mathcal{T}(S)$.
- Since u_1 is tangent to a vertical Bers slice which can be embedded in a fiber of the bundle $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$ (by the map $Z^+ \colon \mathcal{QF}(S) \to \mathcal{CP}(S)$, see § 4.5), one can identify u_1 to a tangent vector to that fiber. Recall that fibers of p are affine spaces modeled on vector spaces

of holomorphic quadratic differentials (see § 4.4), so one can identify u_1 to a holomorphic quadratic differential *i.e.* a cotangent vector to Teichmüller space. We denote it $q_{u_1} \in \mathrm{T}^*\mathcal{T}(S)$.

Proof of Theorem 5.6. Since g is the complex bi-Lagrangian metric associated to the complex bi-Lagrangian structure of Theorem 5.5, the proof of (i), (ii), and (iii) is a direct application of Theorem 2.16, Theorem 2.17, Theorem 3.8 and Theorem 5.5.

Let us now prove (iv). By definition of the complex bi-Lagrangian metric g (cf § 2.7),

$$\begin{aligned} g(u,v) &= \omega(Fu, v) \\ &= \omega(u_1 - u_2, v_1 + v_2) \end{aligned}$$

where $\omega = \omega_G$ is Goldman's symplectic structure. Since both foliations are isotropic for ω, the expression above reduces to $g(u,v) = \omega(u_1, v_2) - \omega(u_2, v_1)$. The conclusion follows from the general fact that $\omega_G(u,v) = i\langle q_u, \mu_v\rangle$ whenever u is vertical and v is horizontal. This fact is a consequence of Theorem 5.2, see [Lou15a, Corollary 6.12] for details. □

5.3 Weil-Petersson metric on Teichmüller space

In this short subsection, we would like to insist that the Weil-Petersson metric on Teichmüller space can be defined purely through symplectic geometry. Indeed, it is sufficient to know that quasi-Fuchsian space is a complex bi-Lagrangian manifold in order to define the Weil-Petersson metric as $-i$ times the restriction of the complex bi-Lagrangian metric to the Fuchsian slice, according to Theorem 5.6. This definition of the Weil-Petersson metric is arguably simpler than the classical definition using the Poincaré metric (see § 4.3), which requires the celebrated but hard uniformization theorem[17]. Let us recap how one argues that quasi-Fuchsian space is a complex bi-Lagrangian manifold:

- $\mathcal{QF}(S)$ is a complex symplectic manifold as an open subset of the smooth locus of the character variety $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$, which enjoys a natural complex symplectic structure by Goldman's algebraic construction (see § 5.1).
- The fact that the two transverse foliations of $\mathcal{QF}(S)$ by horizontal and vertical Bers slices are Lagrangian is part of Theorem 5.5, but it can also be seen as a weaker version of Lemma 5.7.

5.4 Affine bundle structure of $\mathcal{CP}(S)$

One of the striking features of the deformation space of complex projective structures $\mathcal{CP}(S)$ is that the "forgetful projection" $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$ is a holomorphic affine bundle, with underlying vector bundle $\mathrm{T}^*\mathcal{T}(S)$ *i.e.* the holomorphic cotangent bundle to Teichmüller space. The affine structure in the fibers of $p^{-1}(X)$ is not obvious: it is constructed from a differential operator called the Schwarzian derivative. Refer to § 4.4 for details on these facts. We prove in Theorem 5.8 below that this affine structure can instead be defined easily using symplectic geometry: is just the Bott affine structure in the leaves of a Lagrangian foliation as in Theorem 1.12.

Lemma 5.7. *The forgetful projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$ is a holomorphic Lagrangian fibration.*

[17] On the other hand, we define the Weil-Petersson metric on the Fuchsian space $\mathcal{F}(S)$ (or the Fricke-Klein deformation space) rather than on Teichmüller space $\mathcal{T}(S)$. Of course, uniformization is required if one wishes to identify $\mathcal{T}(S)$ to $\mathcal{F}(S)$. Let us mention that there also exists a "uniformization-free" definition of the Weil-Petersson metric on $\mathcal{F}(S)$ purely in terms of Riemannian geometry, see [Tro92, §2.6].

This lemma is a direct consequence of Theorem 5.2 (see [Lou15a, Corollary 6.11]) but it was formerly known: it is a consequence of Kawai's rather involved work [Kaw96], it was also proven directly by Goldman [Gol04] with a concise algebraic argument, and recovered independently by Steven Kerckhoff in unpublished work.

Theorem 5.8. *The complex affine structure in the fibers of the projection* $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$ *constructed via the Schwarzian derivative coincides with the natural complex affine structure in the leaves of a complex Lagrangian foliation.*

The proof of Theorem 5.8 is a straightforward consequence of Theorem 1.13 and Theorem 5.2:

Proof. Choose any holomorphic section $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ to the projection $p : \mathcal{CP}(S) \to \mathcal{T}(S)$, for instance a Bers section, and denote by $\tau_\sigma\colon \mathcal{CP}(S) \to \mathrm{T}^*\mathcal{T}(S)$ the associated isomorphism of affine bundles over $\mathcal{T}(S)$ given by the Schwarzian parametrization using σ as the "zero section". By definition, the restriction of τ^σ to any fiber $p^{-1}(X)$ is an isomorphism of complex affine spaces ${\tau^\sigma}_{|p^{-1}(X)}\colon p^{-1}(X) \to \mathrm{T}^*_X\mathcal{T}(S)$. On the other hand, if σ was chosen among Lagrangian sections *e.g.* Bers sections, then the isomorphism $\tau_\sigma\colon \mathcal{CP}(S) \to \mathrm{T}^*\mathcal{T}(S)$ is additionally a complex symplectomorphism by Theorem 5.2. In particular it is an isomorphism of complex Lagrangian foliations, so that the map ${\tau^\sigma}_{|p^{-1}(X)}$ identifies the complex affine structures of $p^{-1}(X)$ and $\mathrm{T}^*_X\mathcal{T}(S)$ given by their respective Bott connections. However, by Theorem 1.13, the Bott affine structure in $\mathrm{T}^*_X\mathcal{T}(S)$ coincides with its affine structure as a vector space. The conclusion follows. □

5.5 Affine structures on Teichmüller space

It is well-known that Teichmüller space is a Stein manifold. In fact, any Bers embedding $b_{\overline{X}}\colon \mathcal{T}(S) \to H^0(\overline{X}, K^2_{\overline{X}})$ (cf § 4.5) is a holomorphic embedding of $\mathcal{T}(S)$ in a complex vector space of the same dimension, and its image is a bounded domain of holomorphy [BE64]. In particular, Bers embeddings define a family of (incomplete) complex affine structures on $\mathcal{T}(S)$ (in the sense of Definition 1.4), parametrized by $X \in \mathcal{T}(S)$. Remarkably, this family of affine structures is precisely the one predicted by Corollary 3.9:

Theorem 5.9. *The family of affine structures on* $\mathcal{T}(S)$ *parametrized by* $X \in \mathcal{T}(S)$ *of Corollary 3.9 is equal to the family of affine structures on* $\mathcal{T}(S)$ *provided by the Bers embeddings* $b_{\overline{X}}$, $X \in \mathcal{T}(S)$.

Observe that Corollary 3.9 does apply in this situation because we know that the complex symplectic structure $\omega^c_{\mathrm{WP}} = \omega_G$ exists everywhere in the canonical complexification $\mathcal{T}(S)\times\overline{\mathcal{T}(S)} \approx \mathcal{QF}(S)$.

Proof. The map $Z^+\colon \mathcal{QF}(S) \to \mathcal{CP}(S)$ defined in § 4.5 is a complex symplectomorphism which sends the vertical foliation of $\mathcal{QF}(S)$ by Bers slices to (an open subset of) the foliation of $\mathcal{CP}(S)$ given by the fibers of the projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$. It follows that the Bott affine structure in the leaves of the vertical foliation of $\mathcal{QF}(S)$, *i.e.* the family of affine structures of Corollary 3.9, agrees with the Bott affine structure in the fibers of the projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$. By Theorem 5.8, this also coincides with the affine structure in the fibers of the cotangent bundle $\mathrm{T}^*\mathcal{T}(S)$, which in turn is precisely the family of affine structures given by the Bers embeddings. □

Remark 5.10. Note that any manifold equipped with an affine structure can be locally embedded in a vector space of the same dimension in an essentially unique way. Theorem 5.9 could thus be used to define the Bers embeddings themselves, but more work is required to show that they are injective–it is not enough to know that $\mathcal{T}(S)$ is simply connected, for instance.

5.6 Derivative of the Bers embedding at the origin

Fix a point $X^+ \in \mathcal{T}(S)$ and consider the Bers embedding (10):

$$b_{X^+} : \mathcal{T}(S^-) \to H^0(X^+, K^2_{X^+})$$

as defined in § 4.5. The derivative of this map at a point $X^- \in \mathcal{T}(S^-)$ is a linear map

$$\mathrm{d}b_{X^+|X^-} : \mathrm{T}_{X^-}\mathcal{T}(S^-) \to H^0(X^+, K^2_{X^+}) .$$

Recall that $X \mapsto \overline{X}$ yields an identification $\mathcal{T}(S^-) \approx \overline{\mathcal{T}(S)}$, and that $H^0(X^+, K^2_{X^+})$ can be identified as the complex cotangent space $\mathrm{T}^*_{X^+}\mathcal{T}(S)$. Thus the derivative of the Bers embedding b_{X^+} at $X^- = \overline{X^+}$ may be seen as a complex antilinear map

$$\mathrm{d}b_{X^+|\overline{X^+}} : \mathrm{T}_{X^+}\mathcal{T}(S) \to \mathrm{T}^*_{X^+}\mathcal{T}(S) . \tag{12}$$

Theorem 5.11. *The derivative of the Bers embedding at the origin (12) is equal to -1/2 times the musical isomorphism induced by the Weil-Petersson metric:*

$$\mathrm{d}b_{X^+|\overline{X^+}} = -\frac{1}{2}\flat$$

where $\flat$ is the musical isomorphism induced by the Weil-Petersson Hermitian metric h_{WP}:

$$\begin{aligned}\flat :\ \mathrm{T}_{X^+}\mathcal{T}(S) &\to \mathrm{T}^*_{X^+}\mathcal{T}(S)\\ v &\mapsto v^\flat := h_{\mathrm{WP}}(\cdot, v)\ .\end{aligned}$$

Note that Theorem 5.11 is by no means a new result. The derivative of the Bers embedding at the origin is a standard calculation in Teichmüller theory that relies on the following *self-reproducing formula* ([Ber66], [Gar87, §5.7]). Let φ be a holomorphic function on the upper half-plane $\mathbb{H} \subset \mathbb{C}$, denote by $\mathbb{L} \subset \mathbb{C}$ the lower half-plane and let $|\sigma| = \frac{|\mathrm{d}w|^2}{\mathrm{Im}(w)^2}$ denote the area density of the Poincaré metric on $\mathbb{L}$. Then

$$\varphi(z) = \frac{12}{\pi}\int_{\mathbb{L}} \frac{\varphi(\overline{w})}{(w-z)^4}|\sigma| .$$

An concise exposition of this calculation can be found in [McM00, proof of Theorem 7.1]. We thank Curtis McMullen for pointing this out to us and the fact that a much stronger result is proved in [Wol90]. Let us emphasize that we recover the result using symplectic geometry instead of complex analysis, much in the spirit of this section. Specifically, we shall prove Theorem 5.11 using the complex bi-Lagrangian metric g on quasi-Fuchsian space introduced in Theorem 5.6.

Proof of Theorem 5.11. Let v be a tangent vector to $\mathcal{T}(S)$ at X^+, denote by $\overline{v}$ the corresponding tangent vector to $\mathcal{T}(S^-)$ at $X^- = \overline{X^+}$. We need to show that for every $\mu \in \mathrm{T}_{X^+}\mathcal{T}(S)$,

$$\langle (b_{X^+})_*\overline{v}, \mu\rangle = -\frac{1}{2}h_{\mathrm{WP}}(\mu, v) .$$

Let $\rho = QF(X^+, X^-) \in \mathcal{F}(S) \subset \mathcal{QF}(S)$ and consider the following tangent vectors:

$$\begin{aligned} a &= (\mu, \overline{\mu}) \in \mathrm{T}_{(X^+,X^-)}\mathcal{T}(S^+)\times\mathcal{T}(S^-) \qquad & b &= (v, \overline{v}) \in \mathrm{T}_{(X^+,X^-)}\mathcal{T}(S^+)\times\mathcal{T}(S^-)\\ u &= (QF)_* a \in \mathrm{T}_\rho\mathcal{F}(S) \subset \mathrm{T}_\rho\mathcal{QF}(S) \qquad & v &= (QF)_* b \in \mathrm{T}_\rho\mathcal{F}(S) \subset \mathrm{T}_\rho\mathcal{QF}(S)\ .\end{aligned}$$

Using the notations introduced below Theorem 5.6, observe that we have by construction:

$$\begin{aligned} u_1 &= (QF)_*(0, \overline{\mu}) \qquad & q_{u_1} &= (b_{X^+})_*\overline{\mu} \qquad & v_1 &= (QF)_*(0, \overline{v}) \qquad & q_{v_1} &= (b_{X^+})_*\overline{v}\\ u_2 &= (QF)_*(\mu, 0) & \mu_{u_2} &= \mu & v_2 &= (QF)_*(v, 0) & \mu_{v_2} &= v\end{aligned}$$

By Theorem 5.6 (iv), one can thus express $g(u,v)$ as:

$$g(u,v) = i(\langle (b_{X^+})_* \overline{\mu}, v\rangle + \langle (b_{X^+})_* \overline{v}, \mu\rangle) \ . \tag{13}$$

On the other hand, since u and v are both tangent to $\mathcal{F}(S) \subset \mathcal{QF}(S)$ and Theorem 5.6 (i) guarantees that g restricts to $-ig_{\mathrm{WP}}$ on $\mathcal{F}(S)$, we have

$$g(u,v) = -ig_{\mathrm{WP}}(\mu, v) \ . \tag{14}$$

Equating (13) and (14) yields:

$$g_{\mathrm{WP}}(\mu, v) = -\langle (b_{X^+})_* \overline{\mu}, v\rangle - \langle (b_{X^+})_* \overline{v}, \mu\rangle \ .$$

The conclusion follows, recalling that $h_{\mathrm{WP}} = g_{\mathrm{WP}} - i\omega_{\mathrm{WP}}$ and $\omega_{\mathrm{WP}}(\mu, v) = g_{\mathrm{WP}}(i\mu, v)$. □

6 Almost hyper-Hermitian structure in the complexification of a Kähler manifold

In this final section, we construct a natural almost hyper-Hermitian structure in the complexification of a real-analytic Kähler manifold, relying on the canonical complex bi-Lagrangian structure constructed in section 3. This almost hyper-Hermitian structure is unfortunately not integrable in general; in particular it is not the same as the Feix-Kaledin hyper-Kähler structure in the cotangent bundle of a real-analytic Kähler manifold (which we review in § 6.2).

6.1 Almost hyper-Hermitian structures

The algebra of *quaternions* $\mathbb{H}$ is the unital associative algebra over the real numbers generated by three elements i, j, and k satisfying the *quaternionic relations*:

$$i^2 = j^2 = k^2 = -1$$
$$ij = -ji = k$$

$\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$: a generic quaternion is written $q = a + ib + jc + kd$ with $(a,b,c,d) \in \mathbb{R}^4$. We refer to [Lou15b, A.2] for basic notions of quaternionic linear algebra relevant for quaternionic differential geometry.

Definition 6.1. An *almost hyper-Hermitian* structure on M is the data of (g, I, J, K) where:

- g is a Riemannian metric on M.
- I, J, and K are three almost complex structures on M which are compatible with g (*i.e.* g-orthogonal as endomorphisms of $\mathrm{T}M$) and satisfy the quaternionic relations:

$$I^2 = J^2 = K^2 = -\mathbf{1}$$
$$IJ = -JI = K \ .$$

If moreover I, J, and K are parallel for the Levi-Civita connection of g (*integrability condition*) then (g, I, J, K) is called a *hyper-Kähler* structure.

Given an almost hyper-Hermitian manifold (M, g, I, J, K), we denote by ω_I, ω_J, and ω_K the three 2-forms on M defined by $\omega_I = g(I\cdot, \cdot)$, $\omega_J = g(J\cdot, \cdot)$, and $\omega_K = g(K\cdot, \cdot)$.

Remark 6.2. Here is a couple of remarks about Definition 6.1:

- By definition, an almost hyper-Hermitian structure equips a Riemannian manifold (M, g) with an isometric action of the algebra of quaternions $\mathbb{H}$ in its tangent bundle $\mathrm{T}M$. In the language of G-structures, an almost hyper-Hermitian structure is equivalent to a $\mathrm{Sp}(m)$-structure, where $\mathrm{Sp}(m) = \mathrm{U}(m, \mathbb{H})$ is the quaternionic unitary group.
- Some authors use *almost hyper-Kähler* instead of *almost hyper-Hermitian*, e.g. Joyce [Joy00] and Bryant [Bry95]. It seems more consistent to us to call *almost hyper-Kähler* an almost hyper-Hermitian structure whose three Kähler forms ω_I, ω_J, and ω_K are closed, although almost hyper-Kähler structures in this sense are in fact always hyper-Kähler ([Hit87, Lemma 6.8]).

6.2 The Feix-Kaledin hyper-Kähler structure

B. Feix and D. Kaledin independently discovered a canonical hyper-Kähler structure in a neighborhood of the zero section of the holomorphic cotangent bundle of a real-analytic Kähler manifold:

Theorem 6.3 (Feix [Fei99, Fei01], Kaledin [VK99, Kal99])**.** *Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold. There exists a unique*[18] *hyper-Kähler structure (g, I, J, K) in a neighborhood of the zero section in T^*N such that:*

(i) g restricts to g_0 along the zero section.
(ii) I is the standard almost complex structure in T^*N extending I_0.
(iii) $\omega_J + i\omega_K = \omega$ is the canonical complex symplectic form in T^*N (cf Example 1.1)
(iv) The $\mathrm{U}(1)$-action in T^*N by multiplication in the fibers is g-orthogonal.

In the case where $N = \mathbb{C}P^n$ with the Fubini-Study Kähler structure, the Feix-Kaledin metric coincides with the Calabi metric [Cal79], which is a complete hyper-Kähler metric defined everywhere in T^*N. The Eguchi-Hanson metric described in § 6.5.1 is a special case of the Calabi metric when $n = 1$. On the other hand, Feix shows that when N is a compact Riemann surface of genus > 1, the hyper-Kähler metric in a neighborhood of the zero section of T^*N cannot be extended everywhere and is incomplete ([Fei99, Example 5.14]).

The proofs of Feix and Kaledin are both difficult; they are also very different in nature. Let us make a couple of heuristic comments about the proof of Feix [Fei99] because it is relevant to this paper. Feix does not define the hyper-Kähler structure on T^*N directly, instead she constructs the *twistor space* of the hyper-Kähler structure, which is a holomorphic object encoding the hyper-Kähler data (see [HKLR87]). Interestingly, she constructs the twistor space for the complexification N^c rather than the cotangent bundle T^*N[19]. The bi-Lagrangian structure of N^c (see section 3) is key in the in her construction, especially the Bott affine structure in the leaves of the Lagrangian foliations, though she does not use this vocabulary. This initially led us to believe that our construction would recover the same hyper-Kähler structure, but we soon realized that was not the case (see § 6.5).

In this paper, we are interested in hyper-Kähler structures in the complexification N^c of a Kähler manifold rather than in the cotangent bundle T^*N. As we mentioned above, Feix defines a hyper-Kähler structure in N^c, but no theorem is given to characterize the existence and uniqueness of such a structure. There is however at least one proper way to define a "canonical" hyper-Kähler structure in the complexification N^c:

[18] Kaledin [Kal99] states that the hyper-Kähler structure is only unique up to symplectic fiber-wise automorphisms of T^*N. However, unless we are mistaken, there are no such automorphisms besides the identity map.

[19] Feix then recovers T^*N by considering the fiber over 0 instead of the fiber over 1 in the twistor space $Z \to \mathbb{C}P^1$, we refer to [Fei99] for details.

Theorem 6.4. *Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold. Let $(\tilde{g}, \tilde{I}, \tilde{J}, \tilde{K})$ denote the Feix-Kaledin hyper-Kähler structure in a neighborhood V of the zero section of the cotangent bundle* T^*N*. Let $f\colon N \to M$ be a complexification of N.*
There exists a unique hyper-Kähler structure (g, I, J, K) in a sufficiently small neighborhood U of $f(N)$ in M such that there exists an embedding $T\colon U \to V$ (necessarily unique) such that:
(i) T is an isomorphism of hyper-Kähler structures: $T^*(\tilde{g}, \tilde{I}, \tilde{J}, \tilde{K}) = (g, I, J, K)$.
(ii) J is the almost complex structure of M.
(iii) $T \circ f$ is the zero section $N \to \mathrm{T}^*N$.

Proof. The zero section of the cotangent bundle is complex Lagrangian for the canonical complex symplectic structure ω, *a fortiori* it is real Lagrangian for $\omega_{\tilde{J}} = \mathrm{Re}(\omega)$ (cf Theorem 6.3). Since $\omega_{\tilde{J}} = \tilde{g}(\tilde{J}\cdot, \cdot)$, it follows that $\tilde{J}$ sends the tangent space to the zero section to its $\tilde{g}$-orthogonal complement. In particular, T^*N equipped with the complex structure $\tilde{J}$ is a complexification of the zero section. By uniqueness of complexification (cf Theorem 3.3), there exists a unique map $T\colon U \to V$ (for U and V sufficiently small) such that T is holomorphic as map $(U, J) \to (V, \tilde{J})$, and $T \circ f$ is the zero section $N \to \mathrm{T}^*N$. Now just take $g = T^*\tilde{g}$, $I = T^*\tilde{I}$, and $K = T^*\tilde{K}$. □

Definition 6.5. We shall call (g, I, J, K) the *Feix-Kaledin hyper-Kähler structure in the complexification M* (defined in a neighborhood of $N \hookrightarrow M$).

We would like to characterize the Feix-Kaledin hyper-Kähler structure in the complexification as the unique hyper-Kähler *admissible extension* of the Kähler structure in the following sense.

Definition 6.6. Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold. An almost hyper-Hermitian structure (g, I, J, K) in a neighborhood of N in a complexification M is called an *admissible extension* of the Kähler metric off N when:
(i) J is the almost complex structure of M.
(ii) g, I, and ω_I extend g_0, I_0, and ω_0 respectively.
(iii) $\omega_I - i\omega_K$ is the complexification ω_0^c of ω_0.

The following theorem is another straightforward application of analytic continuation (Proposition 3.4), we omit the proof for brevity.

Theorem 6.7. *The Feix-Kaledin hyper-Kähler structure in a neighborhood of N in a complexification M (cf Definition 6.5) is an admissible extension of the Kähler structure off N.*

In the next subsection, we construct another almost hyper-Hermitian admissible extension, but it is typically not integrable. It is unclear to us whether uniqueness of the admissible extension holds amongst hyper-Kähler structures. Note that in the tangent space at points of $N \hookrightarrow N^c$, it is easy to check that uniqueness does indeed hold: it is "just linear algebra" (cf Lemma 6.9).

Question 6.8. Is the Feix-Kaledin hyper-Kähler structure in a sufficiently small neighborhood of N in a complexification M the unique admissible hyper-Kähler extension of the Kähler structure off N?

6.3 Construction of the almost hyper-Hermitian structure

Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold. Let M be a complexification of N, we can assume without loss of generality that $M = N^c = N \times \overline{N}$ is the canonical complexification (cf § 3.2). We are going to construct an almost hyper-Hermitian structure in a neighborhood of N in N^c which is admissible in the sense of Definition 6.6. We start by pointing out that the tensors (g, I, J, K) are uniquely defined in the tangent spaces at points of N inside N^c by the following lemma.

Lemma 6.9. *Let (V, g_0, I_0, ω_0) be a real vector space with a linear Hermitian structure. Denote by $V^c := V \times \overline{V}$ the complexification of (V, I_0), i.e. the real vector space $V \times V$ equipped with the linear complex structure $(I_0, -I_0)$. There exists a unique linear hyper-Hermitian structure (g, I, J, K) in V^c which is admissible in the sense of Definition 6.6.*

The definitions of a linear Hermitian structure and linear hyper-Hermitian structure should be obvious so we do not repeat them, but they may be found in e.g. [Lou15b, A.2]. The proof of Lemma 6.9 is elementary and spared to the reader.

Recall that N^c enjoys a canonical pair of transverse foliations $(\mathcal{F}_1, \mathcal{F}_2)$ (cf § 3.2) and a canonical complex bi-Lagrangian structure (cf Theorem 3.8). We are now ready to state the main theorem of this section.

Theorem 6.10. *There exists a unique almost hyper-Hermitian structure (g, I, J, K) in a sufficiently small neighborhood of N in N^c such that:*

(i) (g, I, J, K) is admissible in the sense of Definition 6.6.

(ii) (g, I, J, K) is parallel with respect to the complex bi-Lagrangian connection of N^c (cf Theorem 3.12) along the foliation $\mathcal{F}_1$.

Moreover, if N is simply connected and ω_0 extends holomorphically throughout N^c, then the almost hyper-Hermitian structure (g, I, J, K) exists throughout N^c.

Theorem 6.10 follows fairly easily Corollary 2.8 and Theorem 2.16:

Proof. Let U be a neighborhood of N in N^c sufficiently small so that:

- The leaves of $\mathcal{F}_1$ are simply connected.
- The complex symplectic form $\omega = \omega_0^c$ extending ω_0 is well-defined.

In particular, the canonical complex bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ is well-defined. Since the tensor fields g, I, J, and K must be parallel with respect to the complex bi-Lagrangian connection ∇ along the vertical foliation $\mathcal{F}_1$, they are completely determined by their values at one point that can be chosen freely in every leaf: their values everywhere else can be obtained by parallel transport along paths contained in the leaves. However the values of g, I, J, and K are uniquely defined at points $p \in N \subset U$ by Lemma 6.9. Thus we have showed uniqueness. Conversely, parallel transport along vertical paths of the tensors g_p, I_p, J_p, and K_p at points $p \in N$ yields tensor fields g, I, J, K that are well-defined in U by virtue of Corollary 2.8 and the simple-connectedness of the leaves. Of course, the linear algebraic identities verified by g_p, I_p, J_p, and K_p are preserved by parallel transport, so that (g, I, J, K) is a almost hyper-Hermitian structure in U. It remains to argue that it is admissible in the sense of Definition 6.6:

(i) J is the almost complex structure of $U \subset N^c$: this is because we know that the almost complex structure of N^c is parallel with respect to the bi-Lagrangian connection by Theorem 2.16.

(ii) g, I, and ω_I extend g_0, I_0, and ω_0 respectively: this is clearly the case by construction.

(iii) $\omega_I - i\omega_K$ is the complexification $\omega = \omega_0^c$ of ω_0: this is because we know that ω is parallel with respect to the bi-Lagrangian connection by Theorem 2.16, therefore the identity $\omega = \omega_I - i\omega_K$ which holds at points $p \in N \subset U$ is preserved under parallel transport.

□

Remark 6.11. The almost hyper-Hermitian structure is typically not integrable (see § 6.5.2 below), therefore it is not the same as the Feix-Kaledin hyper-Kähler structure of Definition 6.5. Note however that it is "2/3 integrable" in the sense that ω_I and ω_K are both closed, since $\omega = \omega_I - i\omega_K$ is the complex symplectic structure of N^c. Let us recall that the integrability of ω_I, ω_J, and ω_K is sufficient for the integrability of the almost hyper-Hermitian structure by [Hit87, Lemma 6.8]. We hypothesize that the flatness of the Kähler metric g_0 is necessary and sufficient for the integrability of the almost hyper-Hermitian structure, in which case it is equal to the Feix-Kaledin hyper-Kähler structure.

6.4 The biquaternionic structure

In this subsection we show that our construction actually yields more than an almost hyper-Hermitian structure. Indeed, the almost hyper-Hermitian structure of Theorem 6.10 naturally combines with the bicomplex Kähler structure associated to the complex bi-Lagrangian structure (cf § 3.4). The resulting "package" is an almost *biquaternionic Hermitian structure*, let us explain this in what follows.

The algebra of *(ordinary) biquaternions* $\mathbb{BH}$ is the unital associative algebra over the real numbers generated by four elements h, i, j, k satisfying the *biquaternionic relations*:

$$\begin{aligned} h^2 = i^2 = j^2 = k^2 = -1 \\ ij = -ji = k \\ hi = ih \quad hj = jh \quad hk = kh \ . \end{aligned} \tag{15}$$

One quickly sees that $\mathbb{BH}$ is an 8-dimensional algebra over $\mathbb{R}$ which can be simply be described as $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ by writing a generic biquaternion $q = q_1 + q_2 h$, where $q_1 = a_1 + ib_1 + jc_1 + kd_1$ and $q_2 = a_2 + ib_2 + jc_2 + kd_2$ are quaternions.

Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold and let M be a complexification of N. We denote by (g, I, J, K) the almost hyper-Hermitian structure of Theorem 6.10. On the other hand, let $H := I_0^c$ denote the holomorphic extension of I_0. Recall that the triple $(H, J, F := HJ)$ is the holomorphic bicomplex structure in (a neighborhood of N in) M associated to the canonical pair of foliations $(\mathcal{F}_1, \mathcal{F}_2)$ by Theorem 3.12. Mind that we denote by H instead of I the almost complex structure I_0^c because I is now a different almost complex structure, and the choice of the letter H is motivated by the following theorem.

Theorem 6.12. *Let (N, g_0, I_0, ω_0) be a real-analytic Kähler manifold and let M be a complexification of N. The Riemannian metric g and the quadruple of almost (para-)complex structures (H, I, J, K) as above define an* almost biquaternionic Hermitian structure *in (a neighborhood of N in) M in the sense that:*

- *H, I, J, K satisfy the biquaternionic relations (15).*
- *H, I, J, K are compatible with g (i.e. g-orthogonal as endomorphisms of $\mathrm{T}M$).*

Proof. Let us only sketch the proof of this theorem for brevity, as we have previously detailed similar arguments. The fact that H, I, J, K satisfy the biquaternionic relations and are compatible with g can be checked directly at points of N inside M because all these tensors are explicit there–it is "just linear algebra". Then we argue that these linear identities are satisfied everywhere because they are preserved by parallel transport. Indeed, the tensor fields g, H, I, J, K are all invariant by parallel transport along the vertical foliation $\mathcal{F}_1$ with respect to the complex bi-Lagrangian connection. For g, I, J, K, this is true by Theorem 6.10. For H, this is because the triple $(g, F = HJ, \omega)$ is the holomorphic bicomplex Kähler structure associated to the canonical bi-Lagrangian structure $(\omega, \mathcal{F}_1, \mathcal{F}_2)$ by Theorem 3.13, it follows that H is parallel with respect to the bi-Lagrangian connection. □

Observe that the algebra of biquaternions $\mathbb{BH}$ contains as subalgebras:

- The algebra of quaternions $\mathbb{H} \approx \mathrm{span}\{1, i, j, k\}$.
- The algebra of bicomplex numbers $\mathbb{BC} \approx \mathrm{span}\{1, h, j, f\}$ where $f := hj$.
- The algebras of *para-quaternions* $\mathbb{P} \approx \mathrm{span}\{1, i, f, g\}$ where $f := hj$ and $g := hk$. This is the unital associative algebra over the real numbers generated by three elements i, f, g satisfying the *para-quaternionic relations*:

$$\begin{aligned} i^2 = -1 \quad f^2 = +1 \quad g^2 = +1 \\ if = -fi = g \ . \end{aligned}$$

Para-quaternions are also called *split-quaternions*, *coquaternions*, and *quaternions of the second kind* in the context of differential geometry [Lib52b, Lib55].

Corresponding to these subalgebras we have an almost hyper-Hermitian structure, an almost bicomplex Hermitian structure and an *almost para-hyper-Hermitian structure* on M. We have already established the presence of the first two: they are respectively the almost hyper-Hermitian structure of Theorem 6.10 and the holomorphic bicomplex Kähler structure of Theorem 3.13. The third one also deserves to be noted:

Corollary 6.13. *Let* (N, g_0, I_0, ω_0) *be a real-analytic Kähler manifold and let* M *be a complexification of* N*. The biquaternionic Hermitian structure* (g, H, I, J, K) *of Theorem 6.12 induces an underlying* almost para-hyper-Hermitian structure (g, I, G, F) *where* $F := HJ$ *and* $G := HK$*.*

Para-hyper-Hermitian structures have been studied under a variety of names such as *para-hyperhermitian* and *para-hyperkähler* [AMT08, DGMY09, Vl11], *hyper-para-Kähler* [Nih04, Mer12], *hypersymplectic*–a terminology coined by Hitchin [Hit90] and followed by many authors, *pseudo-hyperKähler* [DW08, GL11], and *neutral hyperKähler* [Kam99].

6.5 Example: $\mathbb{C}P^1$

Let us compute the Feix-Kaledin hyper-Kähler structure and the almost hyper-Hermitian structure of Theorem 6.10 in the cotangent bundle and complexification of $\mathbb{C}P^1$. We continue using the same notations as in § 3.5.

6.5.1 Feix-Kaledin hyper-Kähler structure

The Feix-Kaledin hyper-Kähler structure in the total space of the cotangent bundle of $\mathbb{C}P^1$ coincides with the metric discovered by Eguchi-Hanson [EH79] and is a special case of *Calabi metric* studied in [Cal79]. However it is typically not expressed in the standard holomorphic coordinates of the cotangent bundle in the mathematics and physics literature. Let us find its expression in such coordinates, it will be instructive to see how to recover it only using the conditions of Theorem 6.3.

Let z be the usual complex coordinate in the affine patch $\mathbb{C} = \mathbb{C}P - \{[1:0]\}$. Let (z, u) be the corresponding coordinates on the holomorphic cotangent bundle $M = \mathrm{T}^*\mathbb{C}P$: a covector $\alpha \in \mathrm{T}^*\mathbb{C}$ having coordinates (z, u) means that $\alpha = u\, dz_{|z}$.

Lemma 6.14. *The Feix-Kaledin hyper-Kähler structure* (g, I, J, K) *in the cotangent bundle* $M = \mathrm{T}^*N$ *of a Riemann surface* N *satisfies* $\omega \wedge \overline{\omega} = 2\omega_I \wedge \omega_I$*, where* ω *is the canonical complex symplectic structure in* $M = \mathrm{T}^*N$*.*

Proof. In any Kähler (in fact Hermitian) manifold (M, g, I, ω_I) of complex dimension n, the identity $\omega_I^n = n!\,\mathrm{vol}_g$ holds. Thus we have here

$$\omega_I \wedge \omega_I = \omega_J \wedge \omega_J = \omega_K \wedge \omega_K = 2\,\mathrm{vol}_g \ .$$

On the other hand, since $\omega = \omega_J + i\omega_K$ (this is a requirement of the Feix-Kaledin hyper-Kähler structure), we have:

$$\omega \wedge \overline{\omega} = \omega_J \wedge \omega_J + \omega_K \wedge \omega_K = 4\,\mathrm{vol}_g \ .$$

The conclusion follows. □

Note that more generally, a similar argument shows that a hyper-Kähler manifold admits a holomorphic volume form, and thus is *Calabi-Yau* (see e.g. [Yau09] for details). Coming back to

$M = \mathrm{T}^*\mathbb{C}P^1$, Lemma 6.14 means that a *Kähler potential* $\varphi\colon M \to \mathbb{R}$ (such that $\frac{i}{2}\partial\overline{\partial}\varphi = \omega_I$) must satisfy the *Monge-Ampère equation*

$$\varphi_{z\bar{z}}\varphi_{u\bar{u}} - \varphi_{u\bar{z}}\varphi_{z\bar{u}} = 1 \ . \tag{16}$$

Let $r\colon M \to \mathbb{R}$ be the function given by $r(\alpha) = \|\alpha\|^2$, where $\|\cdot\|^2$ is the metric in the cotangent bundle induced by g_0, explicitly:

$$r(z,u) = 4u\overline{u}(1+z\overline{z})^2 \ .$$

Since the Feix-Kaledin hyper-Kähler metric g is invariant under the U(1)-action in $M = \mathrm{T}^*M$ which acts transitively in the level sets of r, we can look for a Kähler potential of the form $\varphi(z,u) = y(r)$, where $y\colon [0,+\infty) \to \mathbb{R}$. The Monge-Ampère equation (16) yields the ordinary differential equation

$$8r^2y'y'' + 8r(y')^2 = 1 \ .$$

This ODE is easily solved as it is rewritten $4(Y^2)' = 1$ where $Y = ry'$. The general solution up to an additive constant is

$$y = \sqrt{r+a} - \sqrt{a}\,\mathrm{arcoth}\left(\sqrt{1+\frac{r}{a}}\right)$$

where a is some constant of integration, which we choose $a = 1$ in order to recover the Fubini-Study metric g_0 on the zero section at the end of the calculation. We can now proceed to compute the Kähler form $\omega_I = \frac{i}{2}\partial\overline{\partial}\varphi$, the metric $g = -\omega_I(I\cdot,\cdot)$, etc.

Proposition 6.15. *The Feix-Kaledin hyper-Kähler structure in* $\mathrm{T}^*\mathbb{C}P^1$ *in the coordinates* (z,u) *is given by:*

$$\begin{aligned}
g &= \varphi_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} + \varphi_{u\bar{z}}\,\mathrm{d}u\,\mathrm{d}\bar{z} + \varphi_{z\bar{u}}\,\mathrm{d}z\,\mathrm{d}\bar{u} + \varphi_{u\bar{u}}\,\mathrm{d}u\,\mathrm{d}\bar{u}\\
I &= i\frac{\partial}{\partial z}\otimes \mathrm{d}z - i\frac{\partial}{\partial \bar{z}}\otimes \mathrm{d}\bar{z} + i\frac{\partial}{\partial u}\otimes \mathrm{d}u - i\frac{\partial}{\partial \bar{u}}\otimes \mathrm{d}\bar{u}\\
J &= \left(\varphi_{z\bar{u}}\frac{\partial}{\partial \bar{z}} - \varphi_{z\bar{z}}\frac{\partial}{\partial \bar{u}}\right)\otimes \mathrm{d}z + \left(\varphi_{u\bar{z}}\frac{\partial}{\partial z} - \varphi_{z\bar{z}}\frac{\partial}{\partial u}\right)\otimes \mathrm{d}\bar{z}\\
&\quad + \left(\varphi_{u\bar{u}}\frac{\partial}{\partial \bar{z}} - \varphi_{u\bar{z}}\frac{\partial}{\partial \bar{u}}\right)\otimes \mathrm{d}u + \left(\varphi_{u\bar{u}}\frac{\partial}{\partial z} - \varphi_{z\bar{u}}\frac{\partial}{\partial u}\right)\otimes \mathrm{d}\bar{u}\\
K &= -i\left(\varphi_{z\bar{u}}\frac{\partial}{\partial \bar{z}} - \varphi_{z\bar{z}}\frac{\partial}{\partial \bar{u}}\right)\otimes \mathrm{d}z + i\left(\varphi_{u\bar{z}}\frac{\partial}{\partial z} - \varphi_{z\bar{z}}\frac{\partial}{\partial u}\right)\otimes \mathrm{d}\bar{z}\\
&\quad - i\left(\varphi_{u\bar{u}}\frac{\partial}{\partial \bar{z}} - \varphi_{u\bar{z}}\frac{\partial}{\partial \bar{u}}\right)\otimes \mathrm{d}u + i\left(\varphi_{u\bar{u}}\frac{\partial}{\partial z} - \varphi_{z\bar{u}}\frac{\partial}{\partial u}\right)\otimes \mathrm{d}\bar{u}\\
\omega_I &= \frac{i}{2}\left(\varphi_{z\bar{z}}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} + \varphi_{u\bar{z}}\,\mathrm{d}u\wedge\mathrm{d}\bar{z} + \varphi_{z\bar{u}}\,\mathrm{d}z\wedge\mathrm{d}\bar{u} + \varphi_{u\bar{u}}\,\mathrm{d}u\wedge\mathrm{d}\bar{u}\right)\\
\omega_J &= -\frac{1}{2}\left(\mathrm{d}z\wedge\mathrm{d}u + \mathrm{d}\bar{z}\wedge\mathrm{d}\bar{u}\right)\\
\omega_K &= \frac{i}{2}\left(\mathrm{d}z\wedge\mathrm{d}u - \mathrm{d}\bar{z}\wedge\mathrm{d}\bar{u}\right)
\end{aligned}$$

where:

$$\varphi_{z\bar{z}} = \frac{1+r(1+z\bar{z})}{\sqrt{1+r}(1+z\bar{z})^2} \qquad \varphi_{z\bar{u}} = \frac{2u\bar{z}(1+z\bar{z})}{\sqrt{1+r}}$$

$$\varphi_{u\bar{z}} = \frac{2\bar{u}z(1+z\bar{z})}{\sqrt{1+r}} \qquad \varphi_{u\bar{u}} = \frac{(1+z\bar{z})^2}{\sqrt{1+r}}$$

It is straightforward to check that this hyper-Kähler structure satisfies all the requirements of Theorem 6.3 as expected.

One would like to proceed to compute the Feix-Kaledin hyper-Kähler structure of Definition 6.5 in the complexification of $\mathbb{C}P^1$. This problem seems difficult in general: the natural approach is to find J-holomorphic coordinates in the cotangent bundle extending real coordinates off the zero section, but such a solution of the Newlander-Nirenberg theorem is not explicit in general. An alternative approach is to look for extra symmetries in the case at hand and predict the hyper-Kähler isomorphism $T\colon \mathbb{C}P^1 \times \overline{\mathbb{C}P^1} \to \mathrm{T}^*\mathbb{C}P^1$, as Thomas Hodge does in the case of $\mathbb{H}^2$ in [Hod05, Lemma 3.1.4]. However we were not able to recover an explicit expression of T by using Hodge's ansatz, though it seems likely that a small adjustment of it could work. In any case, we shall see in § 6.5.2 below that on the other hand we are able to explicitly compute the almost hyper-Hermitian structure of Theorem 6.10 in the complexification of $\mathbb{C}P^1$, and that it must be different from this Feix-Kaledin hyper-Kähler structure.

6.5.2 Almost hyper-Hermitian structure

We carry out the explicit computation of the almost hyper-Hermitian structure of Theorem 6.10 in the complexification of $\mathbb{C}P^1$. Most of the work has been done in § 3.5; we resume using the notations introduced there.

Lemma 6.16. *Let $\gamma : [0,1] \to N^c$ be a path contained in a vertical leaf. Denote $\gamma(0) = (z_0, w_0)$ and $\gamma(1) = (z_0, w_1)$. The parallel transport along γ is the linear map $P_\gamma : T_{(z_0,w_0)}N^c \to T_{(z_0,w_1)}N^c$ given by*

$$\begin{aligned} \frac{\partial}{\partial z} &\mapsto \frac{\partial}{\partial z} \\ \frac{\partial}{\partial w} &\mapsto \left(\frac{1+z_0w_1}{1+z_0w_0}\right)^2 \frac{\partial}{\partial w} \;. \end{aligned}$$

NB: In the expression above, P_γ is viewed as a complex linear map between the complexifixed tangent spaces.

Proof. Let $u(t) = a(t)\frac{\partial}{\partial z} + b(t)\frac{\partial}{\partial w}$ be a vector field along γ. Then $u(t)$ is parallel along γ if and only if $\nabla_t u(t) = 0$, which gives the equations:

$$\begin{cases} a'(t) = 0 \\ b'(t) - \frac{2z_0 w'(t)}{1+z_0 w(t)} b(t) = 0 \;. \end{cases}$$

The second equation is easily solved noting that it is an ODE of the form $y'(t) - 2\frac{\alpha'(t)}{\alpha(t)}y(t) = 0$, solutions to this are of the form $y(t) = \left(\frac{\alpha(t)}{\alpha(0)}\right)^2 y_0$. The conclusion follows. □

Note that in particular, P_γ only depends on the endpoints of γ as expected.

We are now ready to compute the almost hyper-Hermitian structure by using vertical parallel transport along the leaves. We compile the results in the following theorem.

Theorem 6.17. *The almost hyper-Hermitian structure in N^c is given by:*

$$
\begin{aligned}
g &= \frac{1}{2}\left[\frac{1}{\left(1+|z|^2\right)^2}\,\mathrm{d}z\,\mathrm{d}\bar{z} + \frac{\left(1+|z|^2\right)^2}{|1+zw|^4}\,\mathrm{d}w\,\mathrm{d}\bar{w}\right]\\
I &= i\left[\overline{\eta}\ \mathrm{d}\bar{w}\otimes\frac{\partial}{\partial z} - \eta\ \mathrm{d}w\otimes\frac{\partial}{\partial \bar{z}} - \frac{1}{\eta}\ \mathrm{d}\bar{z}\otimes\frac{\partial}{\partial w} + \frac{1}{\overline{\eta}}\ \mathrm{d}z\otimes\frac{\partial}{\partial \overline{w}}\right]\\
J &= i\left[\mathrm{d}z\otimes\frac{\partial}{\partial z} - \mathrm{d}\bar{z}\otimes\frac{\partial}{\partial \bar{z}} + \mathrm{d}w\otimes\frac{\partial}{\partial w} - \mathrm{d}\bar{w}\otimes\frac{\partial}{\partial \overline{w}}\right]\\
K &= \overline{\eta}\ \mathrm{d}\bar{w}\otimes\frac{\partial}{\partial z} + \eta\ \mathrm{d}w\otimes\frac{\partial}{\partial \bar{z}} - \frac{1}{\eta}\ \mathrm{d}\bar{z}\otimes\frac{\partial}{\partial w} - \frac{1}{\overline{\eta}}\ \mathrm{d}z\otimes\frac{\partial}{\partial \overline{w}}\\
\omega_I &= \frac{i}{4}\left[\frac{\mathrm{d}z\wedge\mathrm{d}w}{(1+zw)^2} - \frac{\mathrm{d}\bar{z}\wedge\mathrm{d}\bar{w}}{(1+\overline{zw})^2}\right]\\
\omega_J &= \frac{i}{4}\left[\frac{1}{\left(1+|z|^2\right)^2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} + \frac{\left(1+|z|^2\right)^2}{|1+zw|^4}\,\mathrm{d}w\wedge\mathrm{d}\bar{w}\right]\\
\omega_K &= \frac{-1}{4}\left[\frac{\mathrm{d}z\wedge\mathrm{d}w}{(1+zw)^2} + \frac{\mathrm{d}\bar{z}\wedge\mathrm{d}\bar{w}}{(1+\overline{zw})^2}\right]
\end{aligned}
$$

where we have written $\eta = \left(\frac{1+|z|^2}{1+zw}\right)^2$ *in the expressions of* I, J, K.

It is clear from the expressions of ω_I, ω_J, and ω_K that ω_I and ω_K are both closed, in fact $\omega_0^c = \frac{i\ \mathrm{d}z\wedge\mathrm{d}w}{2(1+zw)^2} = \omega_I - i\omega_K$ as expected. One can also check that all the properties of an admissible extension in the sense of Definition 6.6 are indeed satisfied.

On the other hand, we observe that ω_J is not closed, meaning that the hyper-Hermitian structure is not integrable. Thus the Levi-Civita connection of g does not parallelize J, unlike the bi-Lagrangian connection. In particular, the metric g (and hence the hyper-Hermitian structure) differs from the Feix-Kaledin hyper-Kähler structure.

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